

# CROSSED-PRODUCTS EXTENSIONS OF $L_p$ -BOUNDS FOR AMENABLE ACTIONS

ADRIÁN M. GONZÁLEZ-PÉREZ

ABSTRACT. We will extend the transference results in [NR11, CdS15] from the context of noncommutative  $L_p$ -spaces associated with amenable groups to that of noncommutative  $L_p$ -spaces over crossed products of amenable actions. Namely, if  $T_m : L_p(\mathcal{L}G) \rightarrow L_p(\mathcal{L}G)$  is a completely bounded operator, where  $\mathcal{L}G \subset \mathcal{B}(L_2G)$  is the von Neumann algebra of  $G$ , then, we will see that  $\text{Id} \rtimes T_m : L_p(\mathcal{M} \rtimes_\theta G) \rightarrow L_p(\mathcal{M} \rtimes_\theta G)$  is also completely bounded and that

$$\|\text{Id} \rtimes T_m : L_p(\mathcal{M} \rtimes_\theta G) \rightarrow L_p(\mathcal{M} \rtimes_\theta G)\|_{\text{cb}} \leq \|T_m\|_{\text{cb}}$$

provided that  $\theta$  is amenable and trace-preserving. Furthermore, our construction allow to extend  $G$ -equivariant completely bounded operators  $S : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})$  to the crossed-product, so that

$$\|S \rtimes \text{Id} : L_p(\mathcal{M} \rtimes_\theta G) \rightarrow L_p(\mathcal{M} \rtimes_\theta G)\|_{\text{cb}} \leq C^{\frac{1}{p}} \|S\|_{\text{cb}}$$

whenever  $\theta$  is trace-preserving, amenable and its generalized Følner sets satisfy certain accretivity property measured by the constant  $1 \leq C$ . As a corollary we obtain stability results for maximal  $L_p$ -bounds over crossed products. Such results imply the stability under crossed products of the *standard assumptions* used in [GPJP15] to prove a noncommutative generalization of the spectral Hörmander-Mikhlin theorem.

## Introduction

The purpose of this article is to study transference results for operators acting on the  $L_p$ -spaces of crossed products. In order to state and prove our results we will need to recall briefly in this introduction some definitions concerning noncommutative  $L_p$ -spaces, completely bounded operators, crossed products and non-commutative maximal inequalities. We will also provide suitable references for the material here summarized and formulate the main results of the text.

*Noncommutative  $L_p$ -spaces.* Let  $\mathcal{M} \subset \mathcal{B}(H)$  be a von Neumann algebra. If  $\tau_{\mathcal{M}} : \mathcal{M}_+ \rightarrow [0, \infty]$  is a *normal, semifinite and faithful* trace, or a n.s.f. trace in short, then we have a very well understood theory of noncommutative integration. In particular, we can construct a family of Banach spaces, called the *noncommutative  $L_p$ -spaces*. Such spaces are given by completion with respect to the norm  $\|x\|_p = \tau_{\mathcal{M}}(|x|^p)^{1/p}$ , see [PX03] for more information. Naturally, this construction generalizes the classical  $L_p$ -spaces whenever  $\mathcal{M}$  is abelian and  $\tau_{\mathcal{M}}$  is given by integration against a measure. We will denote the  $L_p$ -spaces associated with a trace by  $L_p(\mathcal{M}, \tau_{\mathcal{M}})$ , omitting the dependency on the trace when it can be understood from

---

The author has been partially supported by the ICMAT-Severo Ochoa Excellence Programme SEV-2015-0554 and by the European Research Council consolidator grant 614195.

the context. If  $\mathcal{M} = \mathcal{B}(H)$  and  $\tau_{\mathcal{M}} = \text{Tr}$  is the canonical trace the resulting spaces are called the Schatten classes and denoted by  $S_p(H)$  or  $S_p$  if the dependency on the Hilbert space can be understood from the context. As it is customary, when  $H = \ell_2^n$  we will denote  $S_p(H)$  by  $S_p^n$ .

*Completely bounded maps.* Throughout this article we will use liberally the language of operator spaces. Recall that the category of operator spaces can be defined as that of closed linear subspaces  $E \subset \mathcal{B}(H)$  with morphisms given the, so called, *completely bounded* maps  $\phi : E \rightarrow F$ . I.e. linear maps such that their matrix amplifications  $\text{Id} \otimes \phi : M_n[E] \rightarrow M_n[F]$  are uniformly bounded in  $1 \leq n$ . The spaces  $M_n[E], M_n[F] \subset \mathcal{B}(\ell_2^n \otimes_2 H)$  are just the spaces of  $E$ -valued, resp.  $F$ -valued, matrices with the norm inherited from  $M_n[\mathcal{B}(H)] = \mathcal{B}(\ell_2^n \otimes_2 H)$ . We define the completely bounded norm as

$$\|\phi\|_{\text{cb}} = \|\phi : E \rightarrow F\|_{\text{cb}} = \sup_{1 \leq n} \left\{ \|\text{Id} \otimes \phi : M_n[E] \rightarrow M_n[F]\| \right\}.$$

The space of all completely bounded maps will be denoted by  $\mathcal{CB}(E, F)$  and the term completely bounded will often be shorten to *c.b.* Similarly a map  $\phi$  is completely positive iff all of its matrix amplification are positivity preserving maps. The category of operator spaces is closed under quotients, subsets, interpolation and other operations, see [Pis03, ER00] for more information. We shall also point out that  $L_p(\mathcal{M})$  can be endowed with a canonical operator space structure compatible with interpolation. Such structure is given by interpolation between  $L_\infty(\mathcal{M}) = \mathcal{M} \subset \mathcal{B}(H)$  and  $L_1(\mathcal{M})$ , which will be identified with the predual of the *opposite algebra*, i.e. the algebra  $\mathcal{M}$  with multiplication reversed, see [Pis03, Chapter 7, p. 138-141] for the details.

*The canonical trace.* Let  $G$  be a locally compact and Hausdorff group and let  $\mathcal{LG} \subset \mathcal{B}(L_2 G)$  be the von Neumann algebra generated by the left regular representation  $\lambda$ . Such von Neumann algebra is commonly understood as a noncommutative generalization of the  $L_\infty$ -space over the abelian Pontryagin dual  $\hat{G}$ . The algebra  $\mathcal{LG}$  carries a natural normal, semifinite and faithful (*n.s.f.* in short) weight  $\tau_G : \mathcal{LG} \rightarrow [0, \infty]$ , given by extension of

$$\tau_G(\lambda(f)) = \tau_G\left(\int_G f(g) \lambda_g d\mu(g)\right) = f(e),$$

where  $f \in C_c(G) * C_c(G)$ , see [Ped79, Chapter 7] for the details of the construction of such functional. Such weight coincides with integration against the Haar measure over the dual group if  $G$  is abelian. Observe also that  $\tau_G$  is a tracial weight whenever  $G$  is unimodular. Here, we will work in the context of unimodular groups and the  $L_p$ -spaces associated with the canonical trace will be denoted by  $L_p(\mathcal{LG})$ . Such weight is sometimes called the Plancherel weight since the map  $f \mapsto \lambda(f)$  becomes a unitary isometry  $\lambda : L_2(G) \rightarrow L_2(\mathcal{LG})$ .

*Crossed Products.* Let  $\mathcal{M} \subset \mathcal{B}(H)$  be a von Neumann algebra,  $\tau_{\mathcal{M}} : \mathcal{M}_+ \rightarrow [0, \infty]$  a n.s.f. trace and  $\theta : G \rightarrow \text{Aut}(\mathcal{M})$  a normal and trace preserving action of  $G$ . We define the (spatial) *crossed product*  $\mathcal{M} \rtimes_{\theta} G \subset \mathcal{B}(H \otimes_2 L_2 G)$  as the von Neumann

algebra generated by the images of  $g \mapsto \mathbf{1} \otimes \lambda_g$  and  $\iota : \mathcal{M} \hookrightarrow \mathcal{B}(H \otimes_2 L_2 G)$  given by

$$(\iota x)(\xi)(g) = \theta_{g^{-1}}(x)\xi(g),$$

where  $\xi \in L_2(G; H) \cong H \otimes_2 L_2(G)$ . Observe that  $\mathcal{M} \rtimes_\theta G$  is spanned, after taking weak-\* completions, by binomials of the form  $\iota x \cdot \lambda_g$ . We will usually denote such binomials by  $x \rtimes \lambda_g$ . By [Haa79a, Haa79b] we know that there is a faithful and equivariant operator valued weight  $\mathbb{E}_\mathcal{M} : (\mathcal{M} \rtimes_\theta G)_+ \rightarrow \mathcal{M}_+^\wedge$  generalizing the Plancherel weight when  $\mathcal{M} = \mathbb{C}$ . After composing with  $\tau_\mathcal{M}$  we obtain a n.s.f. weight  $\tau = \tau_\mathcal{M} \circ \mathbb{E}_\mathcal{M}$  that generalizes both  $\tau_G$  over  $\mathbf{1} \otimes \mathcal{L}G \subset \mathcal{M} \rtimes_\theta G$  and  $\tau_\mathcal{M}$  over  $\mathcal{M} \rtimes \mathbf{1} \subset \mathcal{M} \rtimes_\theta G$ . It is easy to see that, since  $\theta$  is trace-preserving,  $\tau$  is a tracial weight whenever  $G$  is unimodular.

*Multipliers.* Our aim in this paper is to transfer the complete boundedness of certain operators acting on  $L_p(\mathcal{M})$  and  $L_p(\mathcal{L}G)$  to  $L_p(\mathcal{M} \rtimes_\theta G)$ . Let  $m \in L_\infty(G)$ . We will denote by  $T_m : L_2(\mathcal{L}G) \rightarrow L_2(\mathcal{L}G)$  the so-called *Fourier multiplier* of *symbol*  $m$ , i.e. the operator given by linear extension of

$$T_m \left( \int_G f(g) \lambda_g d\mu(g) \right) = \int_G m(g) f(g) \lambda_g d\mu(g).$$

It is a trivial consequence of the Plancherel theorem that such operator is bounded in  $L_2(\mathcal{L}G)$ . The boundedness in  $L_p(\mathcal{L}G)$  is a subtle question that has received much attention in the past, both in the commutative setting and in the noncommutative one, see for instance [Pis95a, Har99, JMP14, JMP15] and references within. Fourier multipliers have a close relative in the so called *Schur multipliers*. Let  $a = [a_{i,j}]_{i,j}, b = [b_{i,j}]_{i,j} \subset \mathcal{B}(H)$  be matrices. Their *Schur product* is defined as the cellwise product  $a \bullet b = [a_{i,j} b_{i,j}]_{i,j}$ . Any operator given by multiplication with respect to a fixed  $a$  is called a Schur multiplier. Observe that, since the matrix units  $[e_{i,j}]_{i,j}$  form an orthogonal basis for  $S_2(H)$ , if  $a \in \ell_\infty \otimes \ell_\infty$ , then, the operator  $b \mapsto a \bullet b$  is bounded. Again, determining when a Schur multiplier is bounded in  $S_p(H)$ , for  $p \neq 2$ , is a difficult problem. When  $H = \ell_2(\Gamma)$ , for a discrete group  $\Gamma$ , we can define the *Herz-Schur multipliers* associated with a *symbol*  $m \in \ell_\infty(\Gamma)$  by

$$M_m(e_{g,h}) = m(gh^{-1}) e_{g,h}.$$

Herz-Schur and Fourier multipliers are extremely close object since the former restrict to the later when we restrict the range from  $\mathcal{B}(\ell_2 \Gamma)$  to the subalgebra  $\mathcal{L}\Gamma$ , see the beginning of Section 2 for more information. Recall also that Herz-Schur multipliers can be defined for general locally compact groups just by extension of the pointwise multiplication by the integral kernels  $k \in C_c(G \times G)$  associated to bounded operators, the more or less straightforward details can be consulted at the beginning of [CdLS15] or [LdlS11].

*Maximal Inequalities.* When  $\mathcal{M}$  is a hyperfinite von Neumann algebra and  $E$  is an operator space, there is a notion, due to Pisier, of  $E$ -valued noncommutative  $L_p$ -spaces, see [Pis98]. Indeed, if  $\mathcal{M} = M_n(\mathbb{C})$  is a matrix algebra, then  $S_p^n[E]$  can be defined by operator space interpolation as follows

$$S_p^n[E] := [S_1^n \widehat{\otimes} E, M_n(\mathbb{C}) \otimes_{\min} E]_{\frac{1}{p}},$$

where  $\widehat{\otimes}$  and  $\otimes_{\min}$  are the operator space projective tensor product and minimal tensor product respectively. By hyperfiniteness we can approximate  $L_p(\mathcal{M}; E)$  by

taking direct sums and unions of such finite dimensional  $E$ -valued Schatten classes. This construction was further generalized to the context of QWEP von Neumann algebras by Junge in an unpublished work. A particularly important case of the above construction is that of  $E = \ell_\infty$ . The reason being that the  $L_p(\ell_\infty)$ -norm can be used to define maximal operators in the noncommutative setting. Indeed, recall that if  $(x_n)_n \subset \mathcal{M}$  is a family of noncommuting operators their supremum is an ill-defined object. Nevertheless, thank to the  $\ell_\infty$ -valued noncommutative  $L_p$ -spaces we can speak unambiguously about the  $L_p$ -norm of the supremum just by taking

$$\left\| \sup_{1 \leq n}^+ \{x_n\} \right\|_{L_p(\mathcal{M})} := \|(x_n)_n\|_{L_p(\mathcal{M}; \ell_\infty)},$$

where the  $\sup^+$  in the left hand side is a purely symbolical element. It is also worth recalling that  $L_p(\mathcal{M}; \ell_\infty)$  spaces can be characterized as the elements inside  $\ell_\infty[L_p(\mathcal{M})]$  admitting a factorization of the form  $x_n = \alpha y_n \beta$ , where  $\alpha, \beta \in L_{2p}(\mathcal{M})$  and  $(y_n) \in \ell_\infty \overline{\otimes} \ell_\infty$ . In fact the norm is given by

$$\|(x_n)_n\|_{L_p(\mathcal{M}; \ell_\infty)} = \inf \left\{ \|\alpha\| \sup_n \|y_n\| \|\beta\| \right\},$$

where the infimum is taken over all possible decompositions  $x = \alpha y \beta$ . When  $(x_n)_n$  is a positive element the quantity above can be reduced to

$$\|(x_n)_n\|_{L_p(\mathcal{M}; \ell_\infty)} = \inf_{x_n \leq z} \{ \|z\|_{L_p(\mathcal{M})} \}. \quad (0.1)$$

Indeed the above decomposition allows to generalize the  $L_p(\mathcal{M}; \ell_\infty)$ -spaces to the context of non hyperfinite von Neumann algebras  $\mathcal{M}$ . The noncommutative  $L_p(\ell_\infty)$ -spaces have been used in the past to generalize certain maximal inequalities to the von Neumann algebra setting. Concretely, such technique has been employed in the past to prove noncommutative versions of the Doob maximal inequality for martingales [Jun02] and noncommutative ergodic theorems [JX07]. In [GPJP15], maximals bounds were used to prove a principle of boundedness of Fourier multipliers by maximal operators in the noncommutative setting.

*Summary.* After recalling a few facts from amenable actions in Section 1 we will state and prove our main results in Section 2. In particular, we are going to see that if  $T_m : L_p(\mathcal{L}G) \rightarrow L_p(\mathcal{L}G)$  is a completely bounded Fourier multiplier, then its crossed product extension  $\text{Id} \rtimes T_m$ , given by  $(\text{Id} \rtimes T_m)(x \rtimes \lambda_g) = x \rtimes m(g)\lambda_g$  is completely bounded in  $L_p(\mathcal{M} \rtimes_\theta G)$ , provided that the action  $\theta$  is *amenable*, see [BO08, Section 4.3] or [Zim84, Chapter 4] for a precise definition. Furthermore, our technique yields that

$$\|\text{Id} \rtimes T_m\|_{\mathcal{CB}(L_p(\mathcal{M} \rtimes_\theta G))} \leq \|S_m\|_{\mathcal{CB}(S_p)} \leq \|T_m\|_{\mathcal{CB}(L_p(\mathcal{L}G))}, \quad (0.2)$$

see Corollary 2.3. The techniques involved in the proof of such result are a generalization of the theorems in [NR11] and [CdLS15] from amenable groups to amenable actions. One of the novelties of our approach is that it allows us to transfer, not just Fourier multipliers acting on the  $G$ -component of  $\mathcal{M} \rtimes_\theta G$ , but  $\theta$ -equivariant operators acting of  $\mathcal{M}$ . Indeed, strengthening the amenability of  $\theta$  by imposing an accretivity condition on its generalized “Følner sets” gives a transference result for any completely bounded and  $\theta$ -equivariant operator  $S$  in  $\mathcal{CB}(L_p(\mathcal{M}))$  as follows

$$\|S \rtimes \text{Id}\|_{\mathcal{CB}(L_p(\mathcal{M} \rtimes_\theta G))} \leq C^{\frac{1}{p}} \|S\|_{\mathcal{CB}(L_p(\mathcal{M}))}, \quad (0.3)$$

where  $C \geq 1$  is a constant measuring the accretivity of such sets, see Corollary 2.3. In every example of amenable actions we have worked so far we can build approximating sequences whose accretivity constant is  $C = 1$ . We conjecture that such is the case for all amenable actions. In Section 1 we will state precisely the amenability condition required for our theorems and review briefly the equivalent definitions of amenability for actions.

In Section 3 we will prove an operator-valued extension of the transference results described above. Our extension of the transference results to the  $\ell_\infty$ -valued case allows us to obtain maximal *strong-type maximal inequalities* in crossed products. Concretely, if  $(T_n)_{n \geq 0}$  is a family of completely positive Fourier multipliers over  $L_p(\mathcal{L}G)$  and  $(S_m)_{m \geq 0}$  is a family of completely positive and  $\theta$ -equivariant operators in  $L_p(\mathcal{M})$  and  $\|u\|_p \leq 1$  we have an inequality of the form

$$\begin{aligned} & \left\| \sup_m^+ \sup_n^+ \{ (S_m \rtimes T_n)(u) \} \right\|_{L_p(\mathcal{M} \rtimes_\theta G)} \\ & \lesssim^{(\text{cb})} C^{\frac{1}{p}} \left\| (T_m)_m : L_p(\mathcal{L}G) \rightarrow L_p(\mathcal{L}G; \ell_\infty) \right\|_{\text{cb}} \left\| (S_n)_n : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M}; \ell_\infty) \right\|_{\text{cb}} \end{aligned} \quad (0.4)$$

whenever  $\theta$  has an approximating sequence with accretivity constant  $C$ . Observe that such inequality is a trivial consequence of Fubini type argument when  $\mathcal{M}$  and  $G$  are abelian and the action  $\theta$  is trivial, since  $\mathcal{M} \rtimes_\theta G = \mathcal{M} \overline{\otimes} \mathcal{L}G$ .

As a consequence of those maximal inequalities we obtain that the completely bounded Hardy-Littlewood inequalities, denoted by  $\text{CBHL}_p$ , in [GPJP15] are stable under crossed products if natural invariance conditions are satisfied. Since the rest of the so-called *standard assumptions* in [GPJP15, Definition 2.6] are easily verified for crossed products, we obtain that the standard assumptions are stable under crossed products, see Theorem 3.1. Observe that, for that application, we could have just used the amenability of  $G$  since, by [GPJP15, Remark 2.4/2.5], the standard assumptions imply amenability.

## 1. Amenability of actions

The purpose of this section is to recall a few facts from the theory of amenable actions and provide suitable references. Up to Definition 1.3 all the material here presented is standard and we include it for the sake of completeness.

Let  $(X, \Sigma_X, \nu)$ , or simply  $(X, \nu)$  if the  $\sigma$ -algebra is understood from the context, be a  $\sigma$ -finite measure space. We will say that a group homomorphism  $\theta : G \rightarrow \text{Aut}(X, \nu)$  is action of  $G$  on  $X$  iff the map  $(g, x) \mapsto \theta_g(x) = gx$  is measurable and  $(\theta_g)_*\nu$  and  $\nu$  are mutually absolutely continuous. When  $(\theta_g)_*\nu(E) = \nu(\theta_g E) = \nu(E)$  we will say that  $\theta$  is  $\nu$ -preserving. Throughout this text we will assume that every measure space  $(X, \Sigma_X, \nu)$  is given the Borel structure of an underlying locally compact topological space and that the measure  $\nu$  is regular. Recall also that the action  $\theta$  extends trivially to an action over the functions on  $X$ . We are going to denote it, perhaps ambiguously, by  $\theta_g f(x) = f(\theta_{g^{-1}}x)$ . As usual, if there is no confusion we may just write  $f(g^{-1}x)$  or  $gf$  instead of  $f(\theta_{g^{-1}}x)$ .

A group  $G$  is said to be amenable if there is a translation invariant mean  $m \in L_\infty(G)^*$ , i.e. an element  $m \in L_\infty(G)^*$  such that  $m(f) \geq 0$  for every  $f \geq 0$ ,

$m(\chi_G) = 1$  and  $m(f(g^{-1}\cdot)) = m(f)$  for every  $g \in G$ . We now define a weaker notion of amenability for an action on a von Neumann algebra.

**Definition 1.1.** Let  $\theta : G \rightarrow \text{Aut}(X, \nu)$  be an action, we will say that it is amenable iff there is a (not necessarily normal)  $\theta$ -equivariant conditional expectation  $\mathcal{E} : L_\infty(X) \overline{\otimes} L_\infty(G) \rightarrow L_\infty(X)$ , i.e. a unital, positivity preserving and  $L_\infty(X)$ -linear map, such that

$$\mathcal{E} f(\theta_{g^{-1}}\cdot, g^{-1}\cdot) = \theta_g \mathcal{E}(f).$$

If  $(\mathcal{M}, \varphi)$  is a von Neumann algebra and  $\varphi$  a normal, semifinite and faithful weight, an action  $\theta : G \rightarrow \text{Aut}(\mathcal{M})$  is amenable iff its restriction to the abelian subalgebra  $(\mathcal{Z}(\mathcal{M}), \varphi|_{\mathcal{Z}(\mathcal{M})})$  is amenable, where  $\mathcal{Z}(\mathcal{M})$  denotes the center of  $\mathcal{M}$ .

Observe that, trivially, if  $G$  is amenable all of its actions are amenable, just take  $\mathcal{E} = \text{Id}_{L_\infty(X)} \otimes m$ , for any  $G$ -invariant mean  $m$ . Reciprocally if  $G$  acts amenably on a one-point space then  $G$  is amenable. The flexibility gained is that non-amenable groups may have nontrivial amenable actions. We may also recall that if  $G$  acts amenably in a probability space  $(X, \nu)$  and the measure  $\nu$  is invariant, then, the composition  $\nu \circ \mathcal{E}$  is an invariant mean. The same holds for finite measure spaces with a  $\theta$ -invariant measure.

Like in the case of amenability there are several equivalent characterizations of the property. The definition we have introduced above is not the one that appeared first in the literature. The oldest one, to the knowledge of the author, is that an action  $\theta : G \rightarrow \text{Aut}(X, \nu)$  is amenable iff every affine action on a weak-\* compact convex set *subordinated to*  $\theta$  has a fixed point. A weak-\* compact convex  $G$ -set  $K \subset E^*$  is said to be subordinated to  $\theta : G \rightarrow \text{Aut}(X, \nu)$  iff  $E^*$  can be constructed by tensoring  $L_\infty(X)$  with some dual space  $E_0^*$  and twisting with a 1-cocycle  $\alpha : G \rightarrow \mathfrak{B}(X, \text{Iso}(E_0))$ , where  $\mathfrak{B}(X, A)$  is the space of Borelian functions. A very detailed introduction to such concept can be found in [Zim84, Chapter 4]. Of course, when  $X = \{p\}$ , we get that any affine action of  $G$  in a compact weak-\* closed subset has a fixed point, a condition long known to be equivalent to amenability, see [Pat88]. Amenable actions were introduced in the pioneering works of Zimmer, see [Zim77, Zim78a, Zim78b, Zim78c] following earlier results of Furstenberg [Fur73]. The equivalence with the definition given here was proved in [AEG94]. We shall also point out that the notion of amenability stated here is sometimes referred to as Zimmer-amenability. It shall not be confused with the very different notion of  $X$  admitting a  $\theta$ -invariant mean  $m \in L_\infty(X)^*$  which is sometimes also referred to as amenability for an action.

It is important to recall that amenability of actions can be defined for continuous actions on topological spaces. Pretty much in the same way in which measurable groups are somehow the same objects as topological groups, see [Var85, Chapter 5:6], topological amenable actions are *the same object* as Borel amenable actions. In order to clarify this we will need the following proposition. Recall that we are going to denote by  $\mathbb{P}(G)$ , the probability measures with the  $\sigma(C_0(G))$ -topology and by  $\mathbb{P}_0(G)$  the subset of all absolutely continuous ones with respect to the Haar measure.

**Proposition 1.1.** *Let  $\theta : G \rightarrow \text{Aut}(X, \nu)$  be an action. It is amenable iff for every  $m \in \mathbb{P}_0(G)$ ,  $\epsilon > 0$  and  $K \subset G$  a compact subset there is a Borel map  $\mu : X \rightarrow \mathbb{P}_0(G)$*

such that

$$\sup_{g \in K} \int_X \|g \mu^x - \mu^{g^x}\|_1 d m(x) < \epsilon, \quad (1.1)$$

where  $g d \mu^x(h) = d \mu^x(g^{-1} h)$ .

Whenever a net  $(\mu_\alpha)_\alpha$  satisfies condition (1.1) for every  $m, \epsilon$  and  $K$  provided that  $\alpha$  is large we will say that  $(\mu_\alpha)_\alpha$  is *asymptotically equivariant*. Observe that the condition in the proposition above is equivalent to the existence of an asymptotically equivariant net. To see that, just denote by  $\mu_{m, \epsilon, K}$  the Borel measurable map in Proposition 1.1 and by  $A$  the net given by all triples  $(m, \epsilon, K)$  with the natural order.

**Proof.** Given any Borel map  $\mu : X \rightarrow \mathbb{P}_0(G)$  we can associate to it a unital, positivity preserving and  $L_\infty(X)$ -linear map  $\mathcal{E}_\mu : L_\infty(X) \bar{\otimes} L_\infty(G) \rightarrow L_\infty(X)$  given by

$$\mathcal{E}_\mu(f)(x) = \int_G f(x, g) d \mu^x(g).$$

Clearly all such maps have norm bounded by  $\|\mathcal{E}_\mu(\mathbf{1})\|_\infty = 1$ . The space of bounded maps  $\mathcal{B}(L_\infty(X \times G), L_\infty(X))$  is a dual Banach space since

$$\begin{aligned} \mathcal{B}(L_\infty(X \times G), L_\infty(X)) &= L_\infty(X) \bar{\otimes} L_\infty(X \times G)^* \\ &= L_1(X)^* \bar{\otimes} L_\infty(X \times G)^* \\ &= (L_1(X) \hat{\otimes} L_\infty(X \times G))^* = L_1(X; L_\infty(X \times G))^*, \end{aligned}$$

and the pairing is given by extension of

$$\langle m \otimes f, \mathcal{E} \rangle = \langle m, \mathcal{E}(f) \rangle = \int_X \mathcal{E}(f) d m.$$

Therefore, by the Banach-Alaoglu compactness theorem, the net  $(\mathcal{E}_{\mu_\alpha})_\alpha$  has a weak-\* accumulation point  $\mathcal{E}$ . Since the subset of all conditional expectations is clearly weak-\* closed,  $\mathcal{E}$  is also a conditional expectation. We have to see that if  $(\mu_\alpha)_\alpha$  is asymptotically invariant, then  $\mathcal{E}$  is equivariant. But that is obvious since we have

$$\begin{aligned} \langle \mathcal{E}_{\mu_\alpha}(\theta_g f) - \theta_g(\mathcal{E}_{\mu_\alpha}(f)), m \rangle &= \int_X \int_G f(g^{-1} x, g^{-1} h) \{d \mu_\alpha^x(g h) - d \mu_\alpha^{g^{-1} h}(h)\} d m(x) \\ &\leq \|f\|_\infty \int_X \|g^{-1} \mu_\alpha^x - \mu_\alpha^{g^{-1} x}\|_1 d m(x) \end{aligned}$$

and for every  $g \in G$  such quantity can be made arbitrarily small.

For the reciprocal we have to use that the space of normal conditional expectations from  $L_\infty(X \times G)$  to  $L_\infty(X)$  is dense inside the set of all conditional expectations with respect to the weak-\* topology. Notice that, by applying the Hahn-Banach theorem in every fibre, normal conditional expectations are in correspondence with measurable maps  $\mu : X \rightarrow \mathbb{P}_0(G)$ . If  $\mathcal{E}$  is an equivariant conditional expectation we have that there is a net  $(\mu_\alpha)$  of Borel maps with  $\mathcal{E}_{\mu_\alpha} \rightarrow \mathcal{E}$  in the weak-\* topology. The net  $\mu_\alpha$  is asymptotically equivariant. After identifying Borel maps  $X \rightarrow \mathbb{P}_0(G)$  with a subset of  $L_\infty(X; L_1 G)$  we have that the weak-\* topology of  $\mathcal{B}(L_\infty(X \times G), L_\infty(X))$  corresponds to the  $\sigma(L_1(X; L_\infty G))$  topology. In particular, for every  $g \in G$  we have that  $g \mu_\alpha^x - \mu_\alpha^{g^x}$  tends to zero in the  $\sigma(L_1(X; L_\infty G))$  topology. In particular 0 is in the  $\sigma(L_1(X; L_\infty G))$ -closed convex hull of the set of all the maps

$$S_g = \{x \mapsto (g \mu^x - \mu^{g^x})\}.$$

It is easily seen that such convex set equals the closure of  $S_g$  in the coarsest linear topology making all maps

$$\mu \mapsto \int_X \|\mu^x\|_1 dm(x)$$

continuous. Taking a sequence in the convex hull of  $S_g$  converging to 0 in such topology gives the claim.  $\square$

We recall now the definition of amenability for topological actions. We will say that an action of  $G$  by homeomorphisms  $\theta : G \rightarrow \text{Homeo}(X)$  is a topological action iff the map  $(g, x) \mapsto \theta_g(x)$  is continuous.

**Definition 1.2.** Let  $X$  be a locally compact topological space and  $\theta$  a topological action. The action is said to be amenable iff there is a net of continuous maps  $\mu_\alpha : X \rightarrow \mathbb{P}(G)$ , such that for every  $g \in G$

$$\lim_\alpha \sup_{x \in X} \|g \mu_\alpha^x - \mu_\alpha^{g x}\|_1 = 0.$$

Similarly, an action of  $G$  in a  $C^*$ -algebra  $\mathcal{A}$  is amenable iff its restriction to the center  $\mathcal{Z}(\mathcal{A})$  is (topologically) amenable.

Observe that, since  $\mathbb{P}(G)$  is a compact, each  $\mu_\alpha$  can be lifted to a continuous function on  $\beta X$ , its Stone-Čech compactification and so we obtain that, by construction, a continuous action  $\theta$  on  $X$  is amenable iff its lift  $\beta\theta$  to  $\beta X$  is amenable.

Such topological definition of amenability appeared in the form above for the first time in [HR00]. In contemporary literature is more common to see amenable actions defined in terms of topological spaces. The topic of topological amenable actions has been researched in connection with exactness for groups, a notion introduced in [KW99], since it was proved in [Oza00] that a discrete group is exact iff it has an amenable action on a compact space. See also [Oza06] for more on amenable actions.

Recall that we can identify continuous functions  $x \in C_c(G; \mathcal{M})$  with elements inside  $\mathcal{M} \rtimes_\theta G$  and that the operator valued weight  $\mathbb{E}_\mathcal{M} : (\mathcal{M} \rtimes_\theta G)_+ \rightarrow \mathcal{M}_+^\wedge$  satisfies that

$$\mathbb{E}_\mathcal{M}[x x^*] = \int_G x(g) x(g)^* d\mu(g), \quad (1.2)$$

for any  $x, y \in \mathcal{M} \rtimes_\theta G$ . When working with  $C^*$  algebraic crossed products like below there is no ambiguity assuming  $\mathcal{A} \subset \mathcal{A}^{**}$  to define  $\mathbb{E}_\mathcal{A}$ . The characterization below is easily seen to be equivalent to amenability.

**Lemma 1.2.** ([BO08, Definition 4.3.1/Lemma 4.3.7]) *An continuous action  $\theta : G \rightarrow \text{Aut}(\mathcal{A})$ , where  $\mathcal{A}$  is a unital  $C^*$ -algebra is amenable iff there is a net  $(x_\alpha)_\alpha \subset C_c(G; \mathcal{Z}(\mathcal{A}))$  of compactly supported functions satisfying*

- (i)  $0 \leq x_\alpha(g)$ .
- (ii)  $\int_G |x_\alpha(g)|^2 d\mu(g) = \mathbf{1}_\mathcal{A}$ .
- (iii)  $\lim_\alpha \mathbb{E}_\mathcal{A}\{((\mathbf{1} \rtimes \lambda_g) x_\alpha - x_\alpha)((\mathbf{1} \rtimes \lambda_g) x_\alpha - x_\alpha)^*\}$   
 $= \lim_\alpha \int_G |\theta_g(x_\alpha(g^{-1}h)) - x_\alpha(h)|^2 d\mu(h) = 0.$



Any such net will be called an approximating sequence.

The following proposition ensures that if  $X$  is the Borel space underlying a compact space and  $\theta$  a continuous action, then  $\theta$  is amenable in the measurable sense iff it is amenable in the topological sense.

**Proposition 1.3** ([BO08, Proposition 5.2.1]). *Let  $X$  be a compact Hausdorff space and  $\theta$  a continuous action of  $G$  on  $X$ . Then  $\theta$  is (topologically) amenable iff we can take a net of asymptotically equivariant Borel maps  $\mu_\alpha : X \rightarrow \mathbb{P}_0(G)$ .*

It rests to see that any measurable action comes from a topological action.

**Theorem 1.4** ([Var85, Theorem 5.7]). *For any measurable action  $\theta$  of  $G$  in  $X$  there is a compact Hausdorff topological space  $Y$ , a continuous action  $\theta_0$  of  $G$  on  $Y$  and a  $\theta_0$ -invariant Borel subset  $E \subset Y$  such that  $X$  and  $E$  are isomorphic as  $G$ -spaces.*

Sometimes the Borel subset  $E$  above can be taken to be closed without loss of generality. One of such situations is when the action preserves a finite measure. Let  $(X, \nu)$  be a finite measure space and the action  $\theta$  of  $G$  be  $\nu$ -preserving. If  $E \subset Y$  is like in the theorem above and  $\iota : X \hookrightarrow E \subset Y$  we have that the finite measure  $\iota_*\nu \in M(Y)$  is Borelian and its support is a closed  $G$ -invariant subset  $\text{supp}[\iota_*\nu] \subset E$ . Restricting to such support amounts to removing a null set of  $X$ , see [AEG94, Lemma 1.3]. Similar results follow for  $\nu$ -preserving actions when  $\nu$  is an infinite regular measure changing closed sets by locally closed sets.

As a corollary of the following discussion we obtain that any action  $\theta : G \rightarrow \text{Aut}(\mathcal{M})$  of  $G$  on a von Neumann algebra  $\mathcal{M}$  is amenable iff it has an approximating sequence  $(x_\alpha)_\alpha \subset C_c(G; \mathcal{Z}(\mathcal{M}))$  as in Lemma 1.2. We introduce now the refinement of amenability of actions that we are going to use through the next subsections.

**Definition 1.3.** Let  $(\mathcal{M}, \tau)$  be a semifinite von Neumann algebra and denote  $(\mathcal{Z}(\mathcal{M}), \tau|_{\mathcal{Z}(\mathcal{M})})$  by  $(L_\infty(X), \nu)$ . We say that the action  $\theta : G \rightarrow \text{Aut}(\mathcal{M})$  has a  $C$ -approximating sequence iff there is a sequence of sets  $F_\alpha \subset X \times G$  such that

$$1 \leq \text{ess sup}_x \mu\{g \in G : (x, g) \in F_\alpha\} \leq C \text{ess inf}_x \mu\{g \in G : (x, g) \in F_\alpha\} < \infty, \quad (1.3)$$

and the elements

$$x_\alpha(x, g) = \frac{\chi_{F_\alpha}(x, g)}{\mu\{g \in G : (x, g) \in F_\alpha\}^{\frac{1}{2}}}$$

form an approximating sequence satisfying (iii) in Lemma 1.2.

Many natural amenable actions, for example that of  $\mathbb{F}_r$  in its hyperbolic boundary, admit a 1-approximating sequence. For instance, we can just take  $F_m$  the set of pairs  $(\omega, \eta) \in \partial\mathbb{F}_r \times \mathbb{F}_r$  such that  $\omega$  is an infinite reduced word and  $\eta$  is one of the  $m$  initial subwords of length less than  $m$ . The existence of  $C$ -approximating sequences is stable under natural operations like tensor product extensions  $\text{Id} \otimes \theta : G \rightarrow \text{Aut}(\mathcal{M} \bar{\otimes} \mathcal{M}_2)$ , diagonal products  $\theta_1 \times \theta_2 : G \rightarrow \text{Aut}(\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2)$  or tensor products  $\theta_1 \otimes \theta_2 : G_1 \times G_2 \rightarrow \text{Aut}(\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2)$ . We conjecture that every amenable action admits  $C$ -approximating sequences with  $C = 1$ .

## 2. An asymptotic embedding

In this section we are going to prove the main result of this article. Observe that if  $\theta : G \rightarrow \text{Aut}(\mathcal{M})$  is an action and  $\mathcal{M} \rtimes_{\theta} G$  is the reduced or spatial crossed product, then, the embedding of  $\mathcal{M} \rtimes_{\theta} G$  into  $\mathcal{B}(H \otimes_2 L_2 G)$  factors through the subalgebra  $\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)$ . Indeed, after identifying kernels  $k$  in  $L_{\infty}(G \times G; \mathcal{M})$  with operators in  $\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)$ , the embedding  $j : \mathcal{M} \rtimes_{\theta} G \rightarrow \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)$  is given by extension of the map sending  $u \in C_c(G; \mathcal{M})$  to the operator with kernel

$$k(g, h) = [\theta_g^{-1}(u(g h^{-1}))]_{g, h \in G}.$$

Let  $T_m : \mathcal{L}G \rightarrow \mathcal{L}G$  be a normal and c.b. Fourier multiplier of symbol  $m$  and denote by  $(\text{Id} \rtimes T_m)$  its crossed product amplification, i.e. the normal operator given by linear extension of the map  $x \rtimes \lambda_g \mapsto m(g)x \rtimes \lambda_g$ . A trivial calculation show that the isometry  $j$  intertwines  $\text{Id} \rtimes T_m$  and  $\text{Id} \otimes M_m$  as shown below

$$\begin{array}{ccc} \mathcal{M} \rtimes_{\theta} G & \xhookrightarrow{j} & \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G) \\ \downarrow \text{Id} \rtimes T_m & & \downarrow \text{Id} \otimes M_m \\ \mathcal{M} \rtimes_{\theta} G & \xhookrightarrow{j} & \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G), \end{array}$$

where  $M_m : \mathcal{B}(L_2 G) \rightarrow \mathcal{B}(L_2 G)$  is the c.b. Herz-Schur multiplier given by

$$M_m([a_{gh}]_{g,h}) = [m(g h^{-1}) a_{gh}]_{g,h}.$$

Similarly, let  $S : \mathcal{M} \rightarrow \mathcal{M}$  be an operator and let us denote by  $S \rtimes \text{Id}$  its crossed product amplification, i.e. the map given by extension of  $x \rtimes \lambda_g \mapsto S(x) \rtimes \lambda_g$ . An straightforward calculation shows that the embedding  $j$  intertwines  $S \rtimes \text{Id}$  and  $S_{\theta}$  as follows

$$\begin{array}{ccc} \mathcal{M} \rtimes_{\theta} G & \xhookrightarrow{j} & \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G) \\ \downarrow S \rtimes \text{Id} & & \downarrow S_{\theta} \\ \mathcal{M} \rtimes_{\theta} G & \xhookrightarrow{j} & \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G), \end{array}$$

where the map  $S_{\theta} : \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G) \rightarrow \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)$  is given by

$$S_{\theta}([x_{gh}]) = [\theta_g^{-1} S \theta_g(x_{gh})]_{g,h \in G}.$$

Therefore, if  $S : \mathcal{M} \rightarrow \mathcal{M}$  is a normal c.b. and  $\theta$ -equivariant operator we obtain that

$$\begin{array}{ccc} \mathcal{M} \rtimes_{\theta} G & \xhookrightarrow{j} & \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G) \\ \downarrow S \rtimes \text{Id} & & \downarrow S \otimes \text{Id} \\ \mathcal{M} \rtimes_{\theta} G & \xhookrightarrow{j} & \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G). \end{array}$$

Observe that, a posteriori, such intertwining identities imply that if  $M_m$  is completely bounded so is  $\text{Id} \rtimes T_m$  and that if  $S : \mathcal{M} \rightarrow \mathcal{M}$  is completely bounded and  $\theta$ -equivariant so is  $S \rtimes \text{Id}$ . It is a well-known result, see [BF91], [CdLS15], that the c.b. norm of the Fourier multiplier  $T_m$  bounds the c.b. norm of the Herz-Schur multiplier  $M_m$ . Summing all up, we obtain the following inequalities

$$\begin{array}{ccccc} \|\text{Id} \rtimes T_m\|_{\text{cb}} & \leq & \|\text{Id} \otimes M_m\|_{\text{cb}} & \leq & \|T_m\|_{\text{cb}} \\ \|S \rtimes \text{Id}\|_{\text{cb}} & \leq & \|S \otimes \text{Id}\|_{\text{cb}} & = & \|S\|_{\text{cb}}. \end{array}$$

The purpose of this section is to generalize such results from the crossed product von Neumann algebra  $\mathcal{M} \rtimes_\theta G$  to its noncommutative  $L_p$ -spaces. The main difficulty stems from the fact that the isometry  $j$  is not trace preserving. In fact, it is easy to see that if  $G$  is a finite group, we have that

$$(\tau_{\mathcal{M}} \otimes \text{Tr})(j \mathbf{1}) = |G| \tau(\mathbf{1}),$$

where  $\tau : (\mathcal{M} \rtimes_\theta G)_+ \rightarrow [0, \infty]$  is the trace extending both  $\tau_{\mathcal{M}}$  and  $\tau_G$ . Therefore  $j$  is unbounded in  $L_1(\mathcal{M} \rtimes_\theta G)$  when  $G$  is discrete and infinite. Similar arguments yield that  $j$  is ill-defined in  $L_1$  when  $G$  is noncompact. A way of circumvent this difficulty is to use amenability to approximate the map  $j$  over compact subsets of  $G$ . This way of proceeding was used by E. Ricard and S. Neuwirth in [NR11], when  $\mathcal{M} = \mathbb{C}$  and  $G$  is a discrete amenable group, to prove that if a Herz-Schur multiplier is completely bounded in  $S_p(L_2G)$ , then so is the Fourier multiplier with the same symbol in  $L_p(\mathcal{L}G)$ . Their result was generalized later by M. Caspers and M. de la Salle in [CdS15] to locally compact and amenable groups. They also proved that amenability is necessary for such theorem, at least for  $4 \leq p$  an even integer. We are going to generalize the transference results from amenable groups to amenable actions and from the  $L_p$ -spaces of group algebras  $\mathcal{L}G$  to the  $L_p$ -spaces of crossed products. The way by which we are going to proceed is to use amenability to approximate  $j$  by a net  $j_p^\alpha : L_p(\mathcal{M} \rtimes_\theta G) \rightarrow L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G))$  of complete contractions such that they are “asymptotically isometric”. Then, we can obtain a complete isometry by taking an ultraproduct of all such maps, getting

$$(j_\alpha)_{\alpha}^{\mathcal{U}} : L_p(\mathcal{M} \rtimes_\theta G) \longrightarrow \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G)).$$

Recall that the ultraproduct above must be understood in the operator space sense, see [ER00, Appendix].

Let us start proving the following lemma.

**Lemma 2.1.** *Let  $(\mathcal{M}, \tau)$ ,  $\theta : G \rightarrow \text{Aut}(\mathcal{M})$  be as above and assume that  $\theta$  is  $\tau$ -preserving and amenable and that  $G$  is unimodular. Let  $(x_\alpha)_\alpha \subset C_c(G; \mathcal{Z}(\mathcal{M}))$  be any approximating net for  $\theta$  and  $X_\alpha \in \mathcal{M} \overline{\otimes} \mathcal{B}(L_2G)$  be*

$$(X_\alpha \xi)(g) = \theta_g^{-1}(x_\alpha(g)) \xi(g),$$

where  $\xi \in L_2(G; H)$ . The maps  $j_p^\alpha : L_p(\mathcal{M} \rtimes_\theta G) \rightarrow L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G))$  given by

$$j_p^\alpha(v) = X_\alpha^{\frac{1}{p}} j(v) X_\alpha^{\frac{1}{p}}$$

satisfy that

$$(i) \quad \|j_p^\alpha : L_p(\mathcal{M} \rtimes_\theta G) \rightarrow L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G))\|_{\text{cb}} \leq 1, \text{ for every } 1 \leq p \leq \infty.$$

$$(ii) \quad \lim_{\alpha} \langle (j_p^\alpha u), (j_{p'}^\alpha v) \rangle = \langle u, v \rangle, \text{ where } \frac{1}{p} + \frac{1}{p'} = 1, \text{ for every } 1 \leq p < \infty.$$

**Proof.** The proof of (i) is trivial when  $p = \infty$ . Proving it for  $p = 1$  and applying interpolation yields the desired result. Let  $u \in L_1(\mathcal{M} \rtimes_\theta G)$ . We can decompose  $u$  as  $u = a b^*$ , with  $\|u\|_2 = \|v\|_2 = \|x\|^{\frac{1}{2}}$ . We have that

$$j_1^\alpha(u) = X_\alpha j(a) j(b)^* X_\alpha = (X_\alpha j(u)) (X_\alpha j(v))^*.$$

But, clearly

$$\|j_1^\alpha(x)\|_{L_1(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G))} \leq \|X_\alpha j(u)\|_{L_2(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G))} \|X_\alpha j(v)\|_{L_2(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G))}.$$

It is trivial to notice that, since  $\tau$  is  $\theta$ -invariant  $L_2(\mathcal{M} \rtimes_\theta G) = L_2(\mathcal{M}) \otimes_2 L_2(G)$  and the isomorphism is given by

$$\langle u, v \rangle_{L_2(\mathcal{M} \rtimes_\theta G)} = \int_G \tau_{\mathcal{M}}(u(g)^* v(g)) d\mu(g),$$

after identifying  $u$  affiliated with  $\mathcal{M} \rtimes_\theta G$  with an  $\mathcal{M}$ -valued function of  $G$  in the natural way. On the other hand, if we denote by  $k(g, h) = \theta_g^{-1}(x_\alpha(g)) \theta_g^{-1}(u(g h^{-1}))$  the kernel of  $X_\alpha j(u)$ , we have that

$$\begin{aligned} \|X_\alpha j(u)\|_{L_2(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G))}^2 &= (\tau_{\mathcal{M}} \otimes \text{Tr}) \left\{ \left[ \int_G k(g, h)^* k(g, h) d\mu(h) \right]_{g,k} \right\} \\ &= \int_G \int_G \tau_{\mathcal{M}} \{ |\theta_g^{-1}(x_\alpha(g)) \theta_g^{-1}(u(g h^{-1}))|^2 \} d\mu(g) d\mu(h) \\ &= \int_G \int_G \tau_{\mathcal{M}} \{ |x_\alpha(g) u(h^{-1})|^2 \} d\mu(g) d\mu(h) \quad (2.1) \end{aligned}$$

$$\begin{aligned} &= \int_G \tau_{\mathcal{M}} \left\{ \left( \int_G |x_\alpha(g)|^2 d\mu(g) \right) |u(h^{-1})|^2 \right\} d\mu(h) \\ &= \int_G \tau_{\mathcal{M}} \{ |u(h)|^2 \} d\mu(h), \quad (2.2) \end{aligned}$$

by using the  $\theta$ -invariance of  $\tau_{\mathcal{M}}$  in (2.1) and Condition (ii) on Lemma 1.2 as well as the unimodularity of  $G$  in (2.1). The same follows for  $v$  and this proves (i).

In order to prove (ii) start by noticing that

$$\begin{aligned} \langle j_p^\alpha(u), j_p^\alpha(v) \rangle &= \int_G \int_G \tau_{\mathcal{M}} \{ \theta_g^{-1}(x_\alpha(g)) \theta_g^{-1}(u(g h^{-1})^* v(g h^{-1})) \theta_h^{-1}(x_\alpha(h)) \} d\mu(g) d\mu(h) \\ &= \int_G \int_G \tau_{\mathcal{M}} \{ \theta_{gh}^{-1}(x_\alpha(gh)) \theta_{gh}^{-1}(u(g)^* v(g)) \theta_h^{-1}(x_\alpha(h)) \} d\mu(g) d\mu(h) \\ &= \int_G \int_G \tau_{\mathcal{M}} \{ x_\alpha(gh) \theta_g(x_\alpha(h)) u(g)^* v(g) \} d\mu(g) d\mu(h) \\ &= \int_G \tau_{\mathcal{M}} \{ u(g)^* v(g) \} d\mu(g) + \int_G \tau_{\mathcal{M}} \{ u(g)^* v(g) A \} d\mu(g), \end{aligned}$$

where  $A$  is just

$$\begin{aligned} A &= \int_G x_\alpha(gh) \theta_{g^{-1}}(x_\alpha(h)) d\mu(h) - \mathbf{1}_{\mathcal{M}} \\ &= \int_G x_\alpha(gh) \theta_{g^{-1}}(x_\alpha(h)) d\mu(h) - \int_G |x_\alpha(g)|^2 d\mu(g) \\ &= \int_G x_\alpha(h) (\theta_{g^{-1}}(x_\alpha(gh)) - x_\alpha(h)) d\mu(h) \\ &\leq \left\| \int_G |x_\alpha(h)|^2 d\mu(h) \right\|_{\mathcal{M}}^{\frac{1}{2}} \left( \int_G |\theta_{g^{-1}}(x_\alpha(gh)) - x_\alpha(h)|^2 d\mu(h) \right)^{\frac{1}{2}} \\ &= \mathbb{E}_{\mathcal{M}} [((\mathbf{1} \rtimes \lambda_{g^{-1}}) x_\alpha - x_\alpha) ((\mathbf{1} \rtimes \lambda_{g^{-1}}) x_\alpha - x_\alpha)^*]^{\frac{1}{2}} \longrightarrow 0, \end{aligned}$$

notice that we have used identity (1.2) in the last step. Using Condition (iii) of Lemma 1.2 and the Dominated Convergence Theorem gives the desired claim.  $\square$

We can now proceed to prove the main theorem of this section.

**Theorem 2.2.** *Let  $(\mathcal{M}, \tau_{\mathcal{M}})$ ,  $G$  and  $\theta : G \rightarrow \text{Aut}(\mathcal{M})$  be as above with  $\theta$  amenable. For any  $1 \leq p < \infty$  we have a completely positive and completely isometric map*

$$L_p(\mathcal{M} \rtimes_{\theta} G) \xrightarrow{j_p} \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)).$$

*The isometry  $j_p$  satisfies that if  $M_m$  and  $T_m$  are the Fourier and Herz-Schur multipliers associated to the symbol  $m$ , then*

$$\begin{array}{ccc} L_p(\mathcal{M} \rtimes_{\theta} G) & \xrightarrow{j_p} & \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)) \\ \downarrow (\text{Id} \rtimes T_m) & & \downarrow (\text{Id} \otimes M_m)^{\mathcal{U}} \\ L_p(\mathcal{M} \rtimes_{\theta} G) & \xrightarrow{j_p} & \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)), \end{array}$$

*Furthermore, if  $\theta$  has a  $C$ -approximating sequence and  $S : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})$  is a completely bounded and  $\theta$ -equivariant operator, then*

$$\begin{array}{ccc} L_p(\mathcal{M} \rtimes_{\theta} G) & \xrightarrow{j_p} & \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)) \\ \downarrow (S \rtimes \text{Id}) & & \downarrow (S_{\alpha})_{\alpha}^{\mathcal{U}} \\ L_p(\mathcal{M} \rtimes_{\theta} G) & \xrightarrow{j_p} & \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)), \end{array}$$

where

$$\|S_{\alpha} : L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)) \rightarrow L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G))\|_{\text{cb}} \leq C^{\frac{1}{p}} \|S : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})\|_{\text{cb}}.$$

**Proof.** Let  $j_p^{\alpha}$  be the maps defined in Lemma 2.1, we define the isometry  $j_p$  just by  $j_p = (j_p^{\alpha})_{\alpha}^{\mathcal{U}}$  for some principal ultrafilter  $\mathcal{U}$ . Such map is completely contractive since each  $j_p^{\alpha}$  is. To prove that it is an isometry notice that, for any von Neumann algebra  $\mathcal{N}$  we have

$$\prod_{\alpha, \mathcal{U}} L_p(\mathcal{N}) \subset \left( \prod_{\alpha, \mathcal{U}} L_{p'}(\mathcal{N}) \right)^*, \quad (2.3)$$

and the embedding is isometric. Indeed, such identity is a consequence, when  $1 < p$ , of the fact that the dual of the ultraproduct is larger than the ultraproduct of the duals, see [Pis03, pp. 59-63, (2.8.8)]. For  $p = 1$ , in addition, we have to use the injectivity of the ultraproduct construction, see [Pis03, pp. 59-63, (2.8.2)], and apply it to the inclusion  $L_1(\mathcal{N}) \subset L_1(\mathcal{N})^{**}$ . With identity (2.3) at hand, we have

that

$$\begin{aligned}
\|j_p x\|_{\Pi_{\alpha, \mathcal{U}} L_p} &= \|j_p x\|_{(\Pi_{\alpha, \mathcal{U}} L_{p'})^*} \\
&= \sup_{\|h\|_{p'} \leq 1} |\langle j_p x, h \rangle| \\
&\geq \sup_{\|y\|_{p'} \leq 1} |\langle j_p x, j_{p'} y \rangle| \\
&= \sup_{\|y\|_{p'} \leq 1} \lim_{\alpha, \mathcal{U}} |\langle j_p^\alpha x, j_{p'}^\alpha y \rangle| = \sup_{\|y\|_{p'} \leq 1} |\langle x, y \rangle| = \|x\|_{L_p}.
\end{aligned}$$

Therefore  $j_p$  is an isometry. The fact that it is a complete isometry follows by similar means.

The intertwining identity concerning  $M_m$  and  $T_m$  is trivial since all of the contractions  $j_p^\alpha$  satisfy that

$$j_p^\alpha (\text{Id} \rtimes T_m) = (\text{Id} \otimes M_m) j_p^\alpha$$

and so does their ultraproduct  $j_p$ . The second intertwining relation is more delicate. The reason is that, if we want  $j_p^\alpha$  to intertwine  $S \rtimes \text{Id}$  and  $S \otimes \text{Id}$ , we need, a priori, to impose  $S$  to be  $\mathcal{M}_\alpha$ -bimodular, where  $\mathcal{M}_\alpha$  is the von Neumann algebra given by

$$\mathcal{M}_\alpha = \{\theta_g^{-1} x_\alpha(g)\}_{g \in G}'' \subset \mathcal{Z}(\mathcal{M}).$$

But such condition is too restrictive. To overcome such difficulty, we will assume that net  $(x_\alpha)_\alpha$  comes from a  $C$ -approximating sequence. Then, for any  $\alpha$  we can define the operator  $Y_\alpha \in \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)$  given by

$$(Y_\alpha \xi)(g) = \begin{cases} \left( P_{\alpha, g}^\perp + P_{\alpha, g} \frac{1}{\theta_g^{-1} x_\alpha(g)} \right) \xi(g) & \text{when } g \in G\text{-supp}[x_\alpha] \\ \xi(g) & \text{otherwise,} \end{cases}$$

where  $P_{\alpha, g} \in \mathcal{Z}(\mathcal{M})$  is the orthogonal projection onto the support of  $x_\alpha(g)$ . Clearly, we have that

$$\|Y_\alpha\|_{\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)} \leq \max_x \{1, \text{ess sup}_x \mu\{g \in G : (x, g) \in F_\alpha\}^{\frac{1}{2}}\} < \infty$$

and since  $Y_\alpha X_\alpha = X_\alpha Y_\alpha = \mathbf{1}_{\mathcal{M}} \otimes P_{G\text{-supp}[x_\alpha]}$  we obtain that

$$j_p^\alpha (S \rtimes \text{Id}) = \underbrace{\text{Ad}_{X_\alpha^{1/p}} (S \otimes \text{Id}) \text{Ad}_{Y_\alpha^{1/p}}}_{S_\alpha} j_p^\alpha,$$

where  $\text{Ad}_S$  is the operator given by  $\text{Ad}_S(T) = S^* T S$ . All that rest to do is to estimate the c.b. norm of  $S_\alpha$ . We have

$$\|S_\alpha\|_{\text{cb}} \leq \|\text{Ad}_{X_\alpha^{1/p}}\|_{\text{cb}} \|S \otimes \text{Id}\|_{\text{cb}} \|\text{Ad}_{Y_\alpha^{1/p}}\|_{\text{cb}}. \quad (2.4)$$

Therefore, if  $\lim_{\alpha, \mathcal{U}} \|\text{Ad}_{X_\alpha^{1/p}}\|_{\text{cb}} \|\text{Ad}_{Y_\alpha^{1/p}}\|_{\text{cb}} < \infty$ , then the result follows. We have that

$$\begin{aligned}
\|\text{Ad}_{X_\alpha^{1/p}}\|_{\text{cb}} &\leq \|X_\alpha\|_{\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)}^{\frac{2}{p}} \leq \left( \text{ess inf}_x \mu\{g \in G : (x, g) \in F_\alpha\} \right)^{-\frac{1}{p}} \\
\|\text{Ad}_{Y_\alpha^{1/p}}\|_{\text{cb}} &\leq \|Y_\alpha\|_{\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)}^{\frac{2}{p}} \leq \max_x \{1, \text{ess sup}_x \mu\{g \in G : (x, g) \in F_\alpha\}\}^{\frac{1}{p}}.
\end{aligned}$$

Using property (1.3) in the definition of  $C$ -approximating sequence gives

$$\|S_\alpha : L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)) \rightarrow L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G))\|_{\text{cb}} \leq C^{\frac{1}{p}} \|S : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})\|_{\text{cb}}$$

and that concludes the proof.  $\square$

As a corollary we obtain that, for any amenable action, if  $M_m$  is a completely bounded Herz-Schur multiplier in  $S_p(L_2G)$  then  $\text{Id} \rtimes T_m$  is c.b. in  $L_p(\mathcal{M} \rtimes_\theta G)$ . But [CdS15, Theorem 4.2] asserts that if  $T_m$  is c.b. in  $L_p(\mathcal{L}G)$ , so is  $M_m$  in  $S_p(L_2G)$ . Therefore, we get that if  $T_m$  is c.b. so is  $\text{Id} \rtimes T_m$ . Similarly, if  $S$  is a  $\theta$ -equivariant c.b. operator over  $L_p(\mathcal{M})$  we have that  $S \rtimes \text{Id}$  is also c.b. The corollary below summarises both statements

**Corollary 2.3.** *Let  $\theta : G \rightarrow \text{Aut}(\mathcal{M})$  be an amenable action and  $G$  an unimodular group, for any  $1 \leq p \leq \infty$ , we have that*

$$\begin{aligned} & \|\text{Id} \rtimes T_m : L_p(\mathcal{M} \rtimes_\theta G) \rightarrow L_p(\mathcal{M} \rtimes_\theta G)\|_{\text{cb}} \\ & \leq \|M_m : S_p(L_2G) \rightarrow S_p(L_2G)\|_{\text{cb}} \\ & \leq \|T_m : L_p(\mathcal{L}G) \rightarrow L_p(\mathcal{L}G)\|_{\text{cb}} \end{aligned} \quad (2.5)$$

Furthermore, if  $S \in \mathcal{CB}(L_p(\mathcal{L}G))$  is  $\theta$ -equivariant and  $\theta$  has a  $C$ -approximating sequence, then

$$\begin{aligned} & \|S \rtimes \text{Id} : L_p(\mathcal{M} \rtimes_\theta G) \rightarrow L_p(\mathcal{M} \rtimes_\theta G)\|_{\text{cb}} \\ & \leq C^{\frac{1}{p}} \|S : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})\|_{\text{cb}}. \end{aligned} \quad (2.6)$$

It is still not known whether the amenability condition for the action is necessary or not for the transference results here presented. Recent results in the context of transference between Schur and Fourier multipliers, see [CdS15], and between groups and subgroups, see [CPPR15, GPS16] suggest that amenability may indeed be necessary. We conjecture the following.

**Conjecture 2.1.** If  $\Gamma$  is a discrete group,  $\theta : G \rightarrow \text{Aut}(\mathcal{M})$  is a trace-preserving action and for some  $p \neq 2$  there is a complete isometry

$$L_p(\mathcal{M} \rtimes_\theta \Gamma) \xrightarrow{j_p} \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(\ell_2\Gamma)),$$

satisfying that

$$j_p(\text{Id} \rtimes T_m) = (\text{Id} \otimes M_m)^{\alpha, \mathcal{U}} j_p,$$

then, the action  $\theta$  is amenable.

Observe that, a priori, it is still not known whether the conjecture above implies that the equality

$$\|\text{Id} \rtimes T_m : L_p(\mathcal{M} \rtimes_\theta \Gamma) \rightarrow L_p(\mathcal{M} \rtimes_\theta \Gamma)\|_{\text{cb}} = \|T_m\|_{\text{cb}} \quad (2.7)$$

holds only for amenable actions. It is also unknown if there are multipliers on  $L_p(\mathcal{L}G)$  for which  $\|T_m\|_{\text{cb}}$  and  $\|\text{Id} \rtimes T_m\|_{\text{cb}}$  are not equal.

### 3. Stability of maximal bounds

Let  $\psi : G \rightarrow \mathbb{R}_+$  be a symmetric and conditionally negative function, see [BdlHV08, Appendix C] for the precise definition. We have that the functions  $e^{-t\psi}$  are of positive type and that they induce a semigroup  $S_t = T_{e^{-t\psi}} : \mathcal{L}G \rightarrow \mathcal{L}G$  of self adjoint, trace preserving and completely positive maps, such semigroups are generally referred to as (symmetric) *Markovian semigroups*. The reader is advised to think of  $(S_t)_{t \geq 0}$  as a noncommutative generalization of the heat semigroup. Such semigroups have attracted much attention in the abelian setting for their possibilities to

generalize Harmonic analysis to more abstract contexts, see [Ste70, VSCC92, SC02]. Spectral multipliers are the operators defined by functional calculus over the infinitesimal generator of the semigroup. In our setting such spectral multipliers are given by Fourier multipliers of the form  $T_{m \circ \psi}$ , where  $m : \mathbb{R}_+ \rightarrow \mathbb{C}$ . In [GPJP15], a noncommutative Hörmander-Mikhlin multiplier theorem was proved generalizing earlier works of Alexopoulos [Ale01, Ale94], Hebisch [Heb92] and Duong-Ouhabaz-Sikora [DOS02]. Such result states that, under certain hypothesis,

$$\|T_{m \circ \psi} : L_p(\mathcal{L}G) \rightarrow L_p(\mathcal{L}G)\|_{\text{cb}} \lesssim_{(p)} \sup_{t \geq 0} \|\eta(\cdot) m(t)\|_{H^{s,\infty}(\mathbb{R}_+)}, \quad \text{for } 1 < p < \infty,$$

where  $H^{s,\infty}(\mathbb{R}_+)$  is a Sobolev space with smoothness order  $s$ , for  $s$  large enough, and  $\eta(z)$  is an analytic function decaying fast at 0 and  $\infty$ , see [GPJP15] for the details. In order to prove such result we introduced an element  $X$  in the extended positive cone of  $\mathcal{L}G$ ,  $(\mathcal{L}G)_+^\wedge$ , see [Haa79a, Haa79b] for the precise definition, as the noncommutative analogue of an invariant metric. We regard the triple  $(\mathcal{L}G, \psi, X)$  as a noncommutative generalization of invariant metric spaces over the dual of the group  $G$ . The reason behind such generalization is that we can understand  $X$  as the unbounded function  $\chi \mapsto d(e, \chi)$ , where  $d : \widehat{G} \times \widehat{G} \rightarrow \mathbb{R}_+$  is an invariant metric, recall that by invariance such function determines the whole metric  $d$ . To prove the Hörmander-Mikhlin theorem above we have to impose certain conditions in  $(\mathcal{L}G, \psi, X)$  which we called the standard assumptions. We recall the definition below.

**Definition 3.1.** ([GPJP15, Definition 2.6]) We say that  $(\mathcal{L}G, \psi, X)$  satisfy the, so called, *standard assumptions* iff

- i) The function  $\Phi_X(t) = \tau(\chi_{[0,t]}(X))$  is *doubling*, i.e.

$$\Phi_X(2t) \leq C \Phi_X(t), \quad \forall 0 \leq t.$$

- ii)  $X$  satisfies the *completely bounded Hardy-Littlewood inequality*, or, CBHL in short, for every  $1 < p < \infty$ , i.e.

$$\|(\mathcal{R}_t)_{t \geq 0} : L_p(\mathcal{L}G) \rightarrow L_p(\mathcal{L}G; L_\infty(\mathbb{R}_+))\|_{\text{cb}} \lesssim_{(p)} 1, \quad (\text{CBHL})$$

where  $\mathcal{R}_t(x) = \Phi_X(t)^{-1} (\chi_{[0,t]}(X) \star x)$  and  $\star$  is the noncommutative analogue of the convolution over  $L_1(\mathcal{L}G)$ , given by  $\lambda(f) \star \lambda(g) = \lambda(fg)$ .

- iii) The Markovian semigroup  $S_t = T_{e^{-t}\psi}$  has  $L_2$ -Gaussian bounds, i.e.

$$\tau\{|\chi_{[r,\infty)}(X)\lambda(e^{-t}\psi)|^2\} \lesssim \frac{1}{\Phi_X(\sqrt{t})^{\frac{1}{2}}} e^{-\beta \frac{r^2}{t}} \quad (L_2\text{GB})$$

Observe that, following our analogy of  $T_{e^{-t}\psi}$  with the heat semigroup,  $\lambda(e^{-t}\psi)$  plays the role of the heat kernel and  $L_2\text{GB}$  is just a form of Gaussian bounds. Similarly, if  $X$  is a classical metric  $\chi_{[0,r]}(X)\Phi^{-1}(r)$  is just the characteristic function of the ball of radius  $r$  over the unit after being normalized in  $L_1$  and the maximal norm of ii) is just the  $L_p$ -norm of the Hardy-Littlewood maximal operator.

The goal of this section is to prove that the assumptions defined above are stable under certain cross-products. Let  $(H, \psi_H, X_H)$  and  $(G, \psi_G, X_G)$  be triples satisfying the standard assumption and let  $\theta : G \rightarrow \text{Aut}(H)$  be a  $\mu_H$ -preserving action. Then,  $K = H \rtimes_\theta G$  is again an unimodular group and it is trivial to check that its Haar measure  $\mu_K$  can be identified with  $\mu_H \otimes \mu_G$ . The action  $\theta$  extends to a normal



and  $\tau_H$ -preserving action on  $\mathcal{L}H$ . Let  $\phi : H \rightarrow \mathbb{C}$  be a function inducing a normal c.b. multiplier  $T_\phi$  over  $\mathcal{L}H$ .  $T_\phi$  is  $\theta$ -equivariant, i.e:  $T_\phi(\theta_g(x)) = \theta_g(T_\phi(x))$ , iff  $\phi$  is  $\theta$ -invariant, i.e.:  $\phi(\theta_g(h)) = \phi(h)$ . Therefore, if  $\phi_1 : H \rightarrow \mathbb{C}$  and  $\phi_2 : G \rightarrow \mathbb{C}$  are functions of positive type, the function  $\phi : K \rightarrow \mathbb{C}$  given by

$$\phi(h, g) = \phi_1(h) \phi_2(g)$$

is also of positive type when  $\phi_1$  is  $\theta$ -invariant. Indeed, let  $\{(h_i, g_i)\}_{i=1}^n \subset K$ , then

$$\begin{aligned} \left[ \phi((h_i, g_i)^{-1} (h_j, g_j)) \right]_{i,j} &= \left[ \phi(\theta_{g_i^{-1}}(h_i^{-1} h_j), g_i^{-1} g_j) \right]_{i,j} \\ &= \left[ \phi_1(h_i^{-1} h_j) \phi_2(g_i^{-1} g_j) \right]_{i,j} \geq 0. \end{aligned} \quad (3.1)$$

The positivity of the matrix in the last line follows from the fact that the Schur product respects positivity. Taking  $\phi_1 = e^{-t\psi_H}$  and  $\phi_2 = e^{-t\psi_G}$  gives that  $\psi : K \rightarrow \mathbb{R}_+$  given by  $\psi(h, g) = \psi_H(h) + \psi_G(g)$  is a conditionally negative length when  $\psi_H$  is  $\theta$ -invariant. The next logical step in order to extend the standard assumptions to crossed products is to find a way of defining operators  $X_1 \rtimes \mathbf{1}, \mathbf{1} \rtimes X_2 \in \mathcal{L}K_+^\wedge$  given  $X_1 \in \mathcal{L}H_+^\wedge$  and  $X_2 \in \mathcal{L}G_+^\wedge$ . Notice that if  $\pi : \mathcal{N} \rightarrow \mathcal{R}$  is any normal  $*$ -homomorphism between von Neumann algebras, then  $\pi$  extends to a normal order-preserving map  $\pi : \mathcal{N}_+^\wedge \rightarrow \mathcal{R}_+^\wedge$ . Therefore, it makes sense to apply the  $*$ -automorphisms  $\theta_g$  to  $X_H$ . We will say that  $X_H$  is  $\theta$  invariant if  $\theta_g(X_H) = X_H$  for every  $g \in G$ . Again, extending the normal inclusions  $\iota_1 : \mathcal{M} \hookrightarrow \mathcal{M} \rtimes_\theta G$  and  $\iota_2 : \mathcal{L}G \rightarrow \mathcal{M} \rtimes_\theta G$  to the extended positive cone gives operators

$$\begin{aligned} X_H^2 \rtimes \mathbf{1} &:= \iota_1(X_H^2) \in \mathcal{L}K_+^\wedge \\ \mathbf{1} \rtimes X_G^2 &:= \iota_2(X_G^2) \in \mathcal{L}K_+^\wedge. \end{aligned}$$

and we define the metric  $X \in \mathcal{L}K_+^\wedge$  by the following equation

$$X^2 = X_H^2 \rtimes \mathbf{1} + \mathbf{1} \rtimes X_G^2.$$

**Theorem 3.1.** *Let  $(H, \psi_H, X_H)$  and  $(G, \psi_G, X_G)$  be triples satisfying the standard assumptions and  $\theta : G \rightarrow \text{Aut}(H)$  be a continuous,  $\mu_H$ -preserving action. Assume that  $\psi_H$  and  $X_H$  are  $\theta$ -invariant. Then,  $(K, \psi, X)$ , defined as above, is also standard.*

In the theorem above it is trivial to prove the  $L_2$ -Gaussian bounds and doublingness of  $\Phi_X$ . The key part are the completely bounded Hardy-Littlewood inequalities. In order to prove that, we are going to use an  $\ell_\infty$ -valued version of Theorem 2.2. Notice that we are not imposing amenability of the action  $\theta$  because the standard assumptions force  $G$  to be amenable, see [GPJP15, Remark 2.5], and hence any action is amenable. The stability result for maximal operators will be the following.

**Theorem 3.2.** *Let  $\mathcal{M}$  be a hyperfinite von Neumann algebra,  $G$  a LCH unimodular group and  $\theta : G \rightarrow \text{Aut}(\mathcal{M})$  a  $\tau_{\mathcal{M}}$ -preserving action admitting a  $C$ -approximating sequence. Let  $(\Omega_1, \nu_1)$  and  $(\Omega_2, \nu_2)$  be measure spaces,  $(T_\omega)_{\omega \in \Omega_1}$  be a family of completely positive Fourier multipliers and  $(S_\omega)_{\omega \in \Omega_2}$  is a family of completely positive and  $\theta$ -invariant operators satisfying that*

$$\begin{aligned} A &= \left\| (T_\omega) : L_p(\mathcal{L}G) \rightarrow L_p(\mathcal{L}G; L_\infty(\Omega_1)) \right\|_{\text{cb}} < \infty \\ B &= \left\| (S_\omega) : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M}; L_\infty(\Omega_2)) \right\|_{\text{cb}} < \infty. \end{aligned} \quad (3.2)$$

Then, we have that

$$\left\| (S_\omega \rtimes T_\zeta)_{(\omega, \zeta)} : L_p(\mathcal{M} \rtimes_\theta G) \rightarrow L_p(\mathcal{M} \rtimes_\theta G; L_\infty(\Omega_1) \otimes_{\min} L_\infty(\Omega_2)) \right\|_{\text{cb}}$$

$$\leq C^{\frac{1}{p}} A B.$$

Observe that, in the abelian case with trivial action  $\theta = \mathbf{1}$ , Theorem 3.2 follows by routinely applying Fubini-type arguments. We obtain the following corollary.

**Corollary 3.3.** *Let  $\mathcal{M}$ ,  $G$ ,  $\theta$ ,  $(T_\omega)_{\omega \in \Omega}$  and  $(S_\omega)_{\omega \in \Omega}$  be like in the previous theorem for some fixed  $(\Omega, \nu)$ . We have that*

$$\|(S_\omega \rtimes T_\omega)_\omega : L_p(\mathcal{M} \rtimes_\theta G) \rightarrow L_p(\mathcal{M} \rtimes_\theta G; L_\infty(\Omega))\|_{\text{cb}} \leq C^{\frac{1}{p}} A B,$$

where  $A$  and  $B$  are defined like in (3.2).

Recall that, since each  $T_\omega$  above is a Fourier multiplier, there is an essentially unique symbol  $m_\omega$  such that  $T_\omega = T_{m_\omega}$ . In order to prove the theorems above we need some preliminary results. We will use the following characterization of boundedness for  $L_\infty$ -valued Schur multipliers whose proof we omit.

**Proposition 3.4.** *Let  $(T_{m_\omega})_{\omega \in \Omega} \subset \mathcal{CB}(L_1(\mathcal{L}G))$ , we have that  $(M_{m_\omega})_\omega : S_p(L_2G) \rightarrow S_p[L_\infty(\Omega)]$  boundedly iff for every  $a \in S_p^k$  and  $(b^\omega)_\omega \in S_{p'}^k[L_1(\Omega)]$  we have that*

$$\left| \int_\Omega \sum_{i,j}^k m_\omega(g_i^{-1} g_j) a_{ij} b_{ij}^\omega d\mu(\omega) \right| \leq K \|a\|_{S_p^k} \|(b^\omega)_\omega\|_{S_{p'}^k[L_1]}. \quad (3.3)$$

Furthermore, the optimal  $K$  satisfies that

$$K = \|(M_{m_\omega})_\omega : S_p \rightarrow S_p[L_\infty(\Omega)]\|.$$

The analogous results for complete norms follows after taking  $a_{ij} \in S_p^m$  and  $b_{ij}^\omega \in S_{p'}^m$  in (3.3)

The following theorem is just a vector-valued extension of Theorem 2.2.

**Theorem 3.5.** *Let  $\mathcal{M}$  be a hyperfinite von Neumann algebra and  $G$ ,  $\theta$  be as above with  $\theta$  amenable. For any  $1 \leq p < \infty$  and any operator space  $E$  we have a complete isometry*

$$L_p(\mathcal{M} \rtimes_\theta G; E) \xrightarrow{j_p} \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G); E).$$

Furthermore, when  $E$  is an operator system  $j_p$  is completely positive.

If  $E = C(X_i)$  is any abelian  $C^*$ -algebra, where  $X_i$  are compact Hausdorff spaces, and  $(T_{m_x})_{x \in X_2}$  is a family of Fourier multipliers in  $L_p(\mathcal{L}G)$ , then the following diagram commute

$$\begin{array}{ccc} L_p(\mathcal{M} \rtimes_\theta G; C(X_1)) & \xrightarrow{j_p} & \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G); C(X_1)) \\ \downarrow (\text{Id} \rtimes T_{m_x})_{x \in X_2} & & \downarrow (\text{Id} \otimes M_{m_x})_{x \in X_2} \\ L_p(\mathcal{M} \rtimes_\theta G; C(X_1 \times X_2)) & \xrightarrow{j_p} & \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G); C(X_1 \times X_2)), \end{array}$$

where  $M_{m_x}$  is the Herz-Schur multiplier associated with the symbol  $m_x$ . Furthermore, if  $\theta$  has a  $C$ -approximating sequence and  $(S_x)_{x \in X_2}$  are  $\theta$ -equivariant operators

in  $L_p(\mathcal{M})$ , then

$$\begin{array}{ccc} L_p(\mathcal{M} \rtimes_{\theta} G; C(X_1)) & \xrightarrow{j_p} & \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G); C(X_1)) \\ \downarrow (S_x \rtimes \text{Id})_{x \in X_2} & & \downarrow (S_x^{\alpha})_{x \in X_2}^{\alpha, \mathcal{U}} \\ L_p(\mathcal{M} \rtimes_{\theta} G; C(X_1 \times X_2)) & \xrightarrow{j_p} & \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G); C(X_1 \times X_2)), \end{array}$$

where  $(S_x^{\alpha})_{x \in X_2}$  satisfies that

$$\begin{aligned} & \left\| (S_x^{\alpha})_{x \in X_2} : L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G); C(X_1)) \rightarrow L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G); C(X_1 \times X_2)) \right\|_{\text{cb}} \\ & \leq C^{\frac{1}{p}} \left\| (S_x)_{x \in X_2} : L_p(\mathcal{M}; C(X_1)) \rightarrow L_p(\mathcal{M}; C(X_1 \times X_2)) \right\|_{\text{cb}}. \end{aligned}$$

Before going into the proof we would like to clarify why we choose  $C(X)$ -valued operators instead of  $L_{\infty}(\Omega)$ -valued, for some measure space  $\Omega$ , if all we care about are maximal bounds. The reason is that, in order to pass from the strong maximal type arguments in Theorem 3.2 to the Corollary 3.3 we need to restrict the maximal operator  $(S_{\omega} \rtimes T_{\zeta})_{(\omega, \zeta)}$  to the diagonal  $\omega = \zeta$ . This requires that the diagonal restriction operator  $m : L_{\infty}(\Omega) \otimes L_{\infty}(\Omega) \rightarrow L_{\infty}(\Omega)$ , given by  $m(f \otimes g) = f g$ , to be completely bounded. That is not the case if we take  $L_{\infty}(\Omega) \overline{\otimes} L_{\infty}(\Omega) = L_{\infty}(\Omega)$ . Nevertheless it holds if we take  $L_{\infty}(\Omega) \otimes_{\min} L_{\infty}(\Omega)$ , which is not a von Neumann algebra.

In order to prove Theorem 3.5 we will need the following well-known lemma, whose proof we omit.

**Lemma 3.6** ([Pis98]). *Let  $\mathcal{M}_1, \mathcal{M}_2$  be hyperfinite von Neumann algebras and  $E$  an operator space. If  $\psi : L_p(\mathcal{M}_1) \rightarrow L_p(\mathcal{M}_2)$  is a completely bounded map, then  $\psi \otimes \text{Id}_E : L_p(\mathcal{M}_1; E) \rightarrow L_p(\mathcal{M}_2; E)$  is completely bounded. Furthermore, if  $E$  is an operator system, the map  $\psi \mapsto \psi \otimes_E$  preserves complete positive maps.*

**Remark 3.2.** When  $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$  is hyperfinite and  $p = 1$ , every map  $\phi$  satisfying that  $\phi \otimes \text{Id}_E$  is bounded for every  $E$  is actually completely bounded, the same follows for  $p = \infty$  when  $\psi$  is normal. For general  $p$ , the maps  $\psi$  satisfying that  $\|\psi \otimes \text{Id}_E : L_p(\mathcal{M}; E) \rightarrow L_p(\mathcal{M}; E)\| < \infty$ , when  $E = \ell_{\infty}$ , are called *regular maps* and were studied in [Pis95b]. Such maps are exactly those which can be expressed as linear combinations of completely positive ones. In the non-hyperfinite case the theorem above is false. Indeed, in [Haa85], Haagerup proved that all the completely bounded maps in  $\mathcal{M}$  are linear combinations of completely positive maps iff  $\mathcal{M}$  is hyperfinite.

**Proof. (of Theorem 3.5)** Let  $(x_{\alpha})_{\alpha}$  be an approximating sequence for the amenable action  $\theta$ . We can construct  $X_{\alpha}$  as in the proof of Theorem 2.2. For each  $j_p^{\alpha}$  by

$$j_p^{\alpha} = (\text{Ad}_{X_{\alpha}^{1/p}} j) \otimes \text{Id}_E$$

and by Lemma 3.6 such maps are complete contractions, i.e.

$$\|j_p^{\alpha} : L_p(\mathcal{M} \rtimes_{\theta} G; E) \rightarrow L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G); E)\|_{\text{cb}} \leq 1.$$

They are also completely positive when  $E$  is an operator system. Let us denote temporarily such maps by  $j_{p,E}^\alpha$ . Now it is enough to prove that

$$\lim_{\alpha, \mathcal{U}} \langle (j_{p, \mathcal{B}(H)}^\alpha x), (j_{p', S_1(H)}^\alpha y) \rangle = \langle x, y \rangle, \quad (3.4)$$

where  $\langle \cdot, \cdot \rangle$  is the duality pairing between  $L_p(\mathcal{M} \rtimes_\theta G; S_1(H))$  and  $L_{p'}(\mathcal{M} \rtimes_\theta G; \mathcal{B}(H))$ . That case suffices since we can always embed  $E$  in a completely isometric way inside  $\mathcal{B}(H)$ . The proof of (3.4) follows like in the scalar case. The identity implies that  $j_p = (j_p^\alpha)_\alpha^\mathcal{U}$  is isometric since we can use that

$$\begin{aligned} \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M}; \mathcal{B}(H)) &\subset \left( \prod_{\alpha, \mathcal{U}} L_{p'}(\mathcal{M}; S_1(H)) \right)^* && \text{when } 1 < p \leq \infty \\ \prod_{\alpha, \mathcal{U}} L_1(\mathcal{M}; \mathcal{B}(H)) &\subset \left( \prod_{\alpha, \mathcal{U}} L_\infty(\mathcal{M}; S_1(H)^{**}) \right)^* && \text{otherwise} \end{aligned}$$

and proceed like in the proof of Theorem 2.2. The commutation identities follow similarly.  $\square$

Theorems 3.5 gives a way of transferring bounds from  $\text{Id} \otimes M$  where  $M$  is a  $C(X)$ -valued Schur multiplier to  $\text{Id} \rtimes T$ , where  $T$  is its associated  $C(K)$ -valued Fourier multiplier. In order to bound the maximal operator given by Schur multipliers  $(\text{Id}_\mathcal{M} \otimes M_{m_\omega})_{\omega \in \Omega}$  we need the following transference result generalizing [CdS15, Theorem 4.2] to the  $L_\infty$ -valued case.

**Theorem 3.7.** *Let  $G$  be a LCH and unimodular group,  $\Omega$  a measure space and  $(T_{m_\omega})_{\omega \in \Omega} \subset \mathcal{CB}(L_1(\mathcal{L}G))$  a family of Fourier multipliers. If  $(M_{m_\omega})_{\omega \in \Omega}$  is the associated family of Schur multipliers then, for every  $1 \leq p \leq \infty$*

$$\begin{aligned} &\| (M_{m_\omega})_{\omega \in \Omega} : S_p(L_2G) \rightarrow S_p(L_2G; L_\infty(\Omega)) \|_{\text{cb}} \\ &\leq \| (T_{m_\omega})_{\omega \in \Omega} : L_p(\mathcal{L}G), L_p(\mathcal{L}G; L_\infty(\Omega)) \|_{\text{cb}}. \end{aligned}$$

**Proof.** Let  $\mu$  be a probability measure over  $\Omega$  such that  $L_1(\Omega, \mu)^* = L_\infty(\Omega)$ , by [CdS15, Lemma 4.1] there is a dense subset of exponents  $1 \leq p \leq \infty$  such that we can choose sequences  $x_n$  and  $y_n$  of norm one elements in  $L_p(\mathcal{L}G)$  and  $L_{p'}(\mathcal{L}G)$  such that

$$\lim_n \langle y_n, T_{m_\omega} x_n \rangle = m(e).$$

Let us define  $z_n = y_n \otimes \chi_\Omega \in L_p(\mathcal{L}G; L_1(\Omega, \mu))$ . Since the  $L_1(\Omega; L_p(\mathcal{L}G))$ -norm is larger than the  $L_p(\mathcal{L}G; L_1(\Omega))$ -norm we obtain that  $\|z_n\|_{L_p(\mathcal{L}G; L_1)} \leq 1$ . Furthermore, if  $(T_{m_\omega})_{\omega \in \Omega}$  is a family of multiplier as in the hypothesis, then

$$\lim_n \langle z_n, T_{m_\omega} x_n \rangle = \int_\Omega m_\omega(e) d\mu(\omega), \quad (3.5)$$

where the pairing  $\langle \cdot, \cdot \rangle$  is the duality pairing between  $L_p(\mathcal{L}G; L_\infty)$  and  $L_{p'}(\mathcal{L}G; L_1)$ . Proving

$$\begin{aligned} &\left| \int_\Omega \sum_{i,j}^k m_\omega(g_i^{-1} g_j) a_{ij} b_{ij}^\omega d\mu(\omega) \right| \\ &\leq \| (T_\omega)_\omega : L_p(\mathcal{L}G) \rightarrow L_p(\mathcal{L}G; L_\infty(\Omega)) \|_{\text{cb}} \|a\|_{S_p^k} \| (b^\omega)_\omega \|_{S_p^k[L_1]} \end{aligned} \quad (3.6)$$

implies, by Proposition 3.3, that

$$\|(M_{m_\omega})_\omega\|_{\mathcal{B}(S_p, S_p(L_\infty))} \leq \|(T_{m_\omega})_\omega : L_p(\mathcal{L}G) \rightarrow L_p(\mathcal{L}G; L_\infty)\|_{\text{cb}} \|a\|_{S_p^k} \|(b^\omega)_\omega\|_{S_p^k[L_1]}.$$

In order to obtain the same bound for the complete norms it is enough to take  $a_{ij} \in S_p^m$  and repeat the calculation. Therefore to prove the claim it suffices to prove (3.6). Pick  $x_n$  and  $z_n$  like in (3.5) and consider  $A_n \in S_p^k[L_p(\mathcal{L}G)]$  and  $B_n^\omega \in S_p^k[L_{p'}(\mathcal{L}G; L_1(\Omega))]$  given by

$$\begin{aligned} A_n &= u^* (a \otimes x_n) u \\ B_n^\omega &= u^* (b^\omega \otimes z_n) u \end{aligned}$$

where  $u$  is the unitary in  $M_k \otimes \mathcal{L}G$  given by

$$u = \begin{pmatrix} \lambda_{g_1} & 0 & \cdots & 0 \\ 0 & \lambda_{g_2} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_{g_k} \end{pmatrix}$$

As a consequence  $\|A_n\|_{S_p^k[L_p(\mathcal{L}G)]} = \|x_n\|_{L_p(\mathcal{L}G)}$  and  $\|B_n\|_{S_{p'}^k[L_{p'}(\mathcal{L}G; L_1)]} \leq \|z_n\|_{L_{p'}(\mathcal{L}G; L_1)}$ .

Observe that  $\lambda_{g_i} T_m(\lambda_{g_i}^* x \lambda_{g_j}) \lambda_{g_j}^* = T_{m_{ij}}(x)$ , where  $m_{ij}(h) = m(g_i^{-1} h g_j)$ , therefore

$$\begin{aligned} & \int_\Omega \sum_{i,j}^k m_\omega(g_i^{-1} g_j) a_{ij} b_{ij}^\omega d\mu(\omega) \\ &= \int_\Omega \sum_{i,j}^k a_{ij} b_{ij}^\omega \lim_n \langle z_n, T_{m_{ij}}^\omega x_n \rangle d\mu(\omega) \\ &= \lim_n \int_\Omega \sum_{i,j}^k a_{ij} b_{ij}^\omega \langle z_n, T_{m_{ij}}^\omega x_n \rangle d\mu(\omega) \\ &= \lim_n \langle (B_n^\omega)_\omega, ((\text{Id} \otimes T_{m_\omega}) A_n)_\omega \rangle \\ &\leq \| (T_{m_\omega})_\omega : L_p(\mathcal{L}G) \rightarrow L_p(\mathcal{L}G; L_\infty) \|_{\text{cb}} \|a\|_{S_p^k} \|(b^\omega)_\omega\|_{S_p^k[L_1]}. \end{aligned}$$

We have used the Dominated Convergence Theorem to exchange the limit and the integral in the second equation, which is justified since the multipliers  $m_\omega$  are uniformly bounded.  $\square$

We can pass to the proof of the strong maximal bounds. Since we are going to reduce the problem to that of tensor product it is convenient to recall the following modification of the result for tensor products, see [GPJP15, Lemma 2.8], whose proof is a trivial consequence of (0.1). We include the proof for the sake of completeness.

**Lemma 3.8.** *Let  $(M_i, \tau_i)$ , for  $i \in \{1, 2\}$  be two hyperfinite von Neumann algebras with n.s.f. traces,  $(\Omega_i, \nu_i)$  two measure spaces and  $(S_\omega)_{\omega \in \Omega_1}$ ,  $(T_\omega)_{\omega \in \Omega_2}$  be families of completely positive operators satisfying that*

$$\begin{aligned} A_1 &:= \| (T_\omega)_{\omega \in \Omega_1} : L_p(\mathcal{M}_1) \rightarrow L_p(\mathcal{M}_1; L_\infty(\Omega_1)) \|_{\text{cb}} < \infty \\ A_2 &:= \| (S_\omega)_{\omega \in \Omega_2} : L_p(\mathcal{M}_2) \rightarrow L_p(\mathcal{M}_2; L_\infty(\Omega_2)) \|_{\text{cb}} < \infty. \end{aligned}$$

Then, we have that

$$\| (R_{\omega, \zeta})_{(\omega, \zeta)} : L_p(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2) \rightarrow L_p(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2; L_\infty(\Omega_1) \otimes_{\min} L_\infty(\Omega_2)) \|_{\text{cb}} \leq A_1 A_2,$$

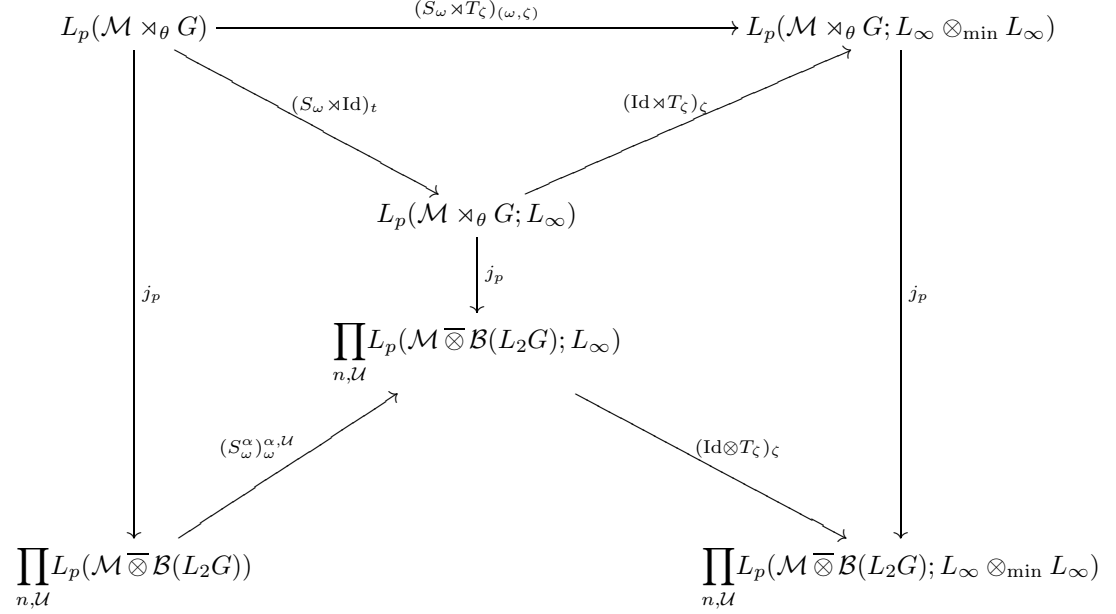


FIGURE 1. Commutative diagram for the proof of Theorem 3.2.

where  $R_{\omega, \zeta} = \text{Ad}_Y (T_\omega \otimes \text{Id}) \text{Ad}_X (\text{Id} \otimes S_\zeta)$  and  $X, Y \in \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$  are self adjoint contractive operators.

**Proof.** Recall that if  $\phi : E \rightarrow F$  is completely bounded, then  $\|\text{Id} \otimes \phi : S_p^n[E] \rightarrow S_p^n[F]\| \leq \|\phi\|_{\text{cb}}$ . As a consequence, the same is true for  $\text{Id} \otimes \phi : L_p(\mathcal{M}; E) \rightarrow L_p(\mathcal{M}; F)$ , when  $\mathcal{M}$  is hyperfinite. We are going to use also that  $L_p(\mathcal{M}_1; L_p(\mathcal{M}_2; L_\infty)) = L_p(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2; L_\infty)$ . By complete positivity of  $(\text{Id} \otimes S_\zeta)$  and (0.1) we have that for every  $x \in L_p(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)$  there is another  $u$  with  $\|u\|_p \leq (1 + \epsilon) B\|x\|_p$ , where  $\epsilon$  can be taken arbitrarily small. Now

$$\begin{aligned} R_{\omega, \zeta}(x) &= \text{Ad}_Y (T_\omega \otimes \text{Id}) \text{Ad}_X (\text{Id} \otimes S_\zeta)(x) \\ &\leq \text{Ad}_Y (T_\omega \otimes \text{Id}) \text{Ad}_X u \end{aligned}$$

and applying the same procedure to  $\text{Ad}_X u$  once again gives the desired identity.  $\square$

**Proof. (of Theorem 3.2).** Recall that for any measure space  $\Omega$ , the algebra  $L_\infty(\Omega)$  is isomorphic to  $C(X)$  where  $X$  is certain compact Hausdorff and disconnected topological space. In order to apply Theorem 3.5 we need to express an element  $\omega \mapsto T_\omega$  inside  $L_\infty(\Omega; \mathcal{CB}(L_p(\mathcal{N})))$  as a  $\mathcal{CB}(L_p(\mathcal{N}))$ -valued function on  $C(X)$ . But, since  $X \subset \text{Ball}(L_\infty(\Omega)^*)$ , we can safely evaluate  $\phi \otimes \text{Id}$ , where  $\phi \in X$ , against  $(T_\omega)_\omega$ . By an application of Theorem 3.5 the diagram in Figure 1 commutes. The  $j_p$  are the complete isometries of Theorem 3.5. The isometries  $j_p$  intertwine  $(S_\omega \rtimes \text{Id})$  with the ultraproduct with respect to  $\mathcal{U}$  in  $\alpha$  of the maps

$$S_\omega^\alpha = \text{Ad}_{X_\alpha^{1/p}} (S_\omega \otimes \text{Id}) \text{Ad}_{Y_\alpha^{1/p}}$$

and so  $(S_\omega \rtimes \text{Id})_{\omega \in \Omega_2}$  is completely bounded (resp. completely positive) if the ultraproduct of such maps is completely bounded (resp. completely positive). But,

since each  $S_\omega$  is c.p. and  $\mathcal{M}$  is hyperfinite that follows by Lemma 3.6. Similarly,  $(\text{Id} \rtimes T_\zeta)_{\zeta \in \Omega_2}$  is completely bounded (resp. completely positive) if  $(\text{Id} \otimes M_\zeta)_{\zeta \in \Omega_2}$  is c.b. (resp. c.p.), where  $M_\zeta$  is the Schur multiplier sharing its symbol with  $T_\zeta$ . By Theorem 3.7  $(\text{Id} \otimes M_\zeta)_\zeta$  is completely bounded. Now, applying Lemma 3.8 gives that  $((\text{Id} \otimes M_\zeta) S_\omega^\alpha)_{(\omega, \zeta)}$  is completely bounded and that finishes the proof.  $\square$

The Corollary 3.3 follows from the Theorem above after applying Lemma 3.9.

**Lemma 3.9.** *Let  $\mathcal{A}$  be an abelian  $C^*$ -algebra and  $(\mathcal{M}, \tau)$  a hyperfinite von Neumann algebra with a n.s.f. trace  $\tau$ , then*

$$\|(\text{Id}_{\mathcal{M}} \otimes m) : L_p(\mathcal{M}; \mathcal{A} \otimes_{\min} \mathcal{A}) \rightarrow L_p(\mathcal{M}; \mathcal{A})\|_{\text{cb}} \leq 1,$$

where  $m : \mathcal{A} \otimes_{\min} \mathcal{A} \rightarrow \mathcal{A}$  is given by  $f \otimes g \mapsto fg$ .

**Proof. (of Corollary 3.3)** Notice that, if  $\mathcal{R}_1 = (S_\omega \rtimes T_\zeta)_{(\omega, \zeta)}$  and  $\mathcal{R}_2 = (S_\omega \rtimes T_\omega)_\omega$ , we have that:

$$\mathcal{R}_2 = (\text{Id}_{\mathcal{M} \rtimes G} \otimes m) \mathcal{R}_1,$$

and applying Lemma 3.9 together with Theorem 3.2 gives the desired result.  $\square$

With that at hand we can pass to prove of the stability under crossed products of the standard assumptions.

**Proof. (of Theorem 3.1)** To prove that  $X$  is doubling we just use that  $X_H^2 \rtimes \mathbf{1}$  and  $\mathbf{1} \rtimes X_G^2$  commute when  $X_H$  is  $\theta$  invariant and therefore:

$$\chi_{[0, r^2]}(X) \leq \chi_{[0, r^2]}(X_H \rtimes \mathbf{1}) \chi_{[0, r^2]}(\mathbf{1} \rtimes X_G).$$

Using that  $\tau_K((x \rtimes \mathbf{1})(\mathbf{1} \rtimes y)) = \tau_H(x) \tau_G(y)$  gives that  $\Phi_X(r) \leq \Phi_{X_H}(r) \Phi_{X_G}(r)$ . Similarly it can be proved that  $\Phi_{X_H}(r) \Phi_{X_G}(r) \leq \Phi_X(2r)$  and therefore  $X$  is doubling. The  $L_2$ GB property is proved similarly. For the CBHL maximal inequalities we just use that

$$\frac{\chi_{[0, r]}(X)}{\Phi_X(r)} \star u \lesssim_{(D_{\Phi_{X_H}}, D_{\Phi_{X_G}})} (\mathcal{R}_r^H \rtimes \mathcal{R}_r^G)(u),$$

where

$$\mathcal{R}_r^H(u) = \frac{\chi_{[0, r]}(X_H)}{\Phi_{X_H}(r)} \star u \quad \text{and} \quad \mathcal{R}_r^G(u) = \frac{\chi_{[0, r]}(X_G)}{\Phi_{X_G}(r)} \star u.$$

The maximal boundedness of  $(\mathcal{R}_r^H \rtimes \mathcal{R}_r^G)_{r \geq 0}$  follows from Corollary 3.3.  $\square$

**Acknowledgement.** The author was informed, through a personal communication, that some of the results here exposed were obtained, in the context of certain amenable groups instead of amenable actions, independently by Quanhua Xu. I am also thankful to Éric Ricard for some discussions concerning the necessity of the amenability condition for the action.

## REFERENCES

- [AEG94] Scot Adams, George A. Elliott, and Thierry Giordano. Amenable actions of groups. *Trans. Amer. Math. Soc.*, 344(2):803–822, 1994.
- [Ale94] G. Alexopoulos. Spectral multipliers on Lie groups of polynomial growth. *Proc. Amer. Math. Soc.*, 120(3):973–979, 1994.
- [Ale01] G. Alexopoulos. Spectral multipliers on discrete groups. *Bulletin of the London Mathematical Society*, 33:417–424, 7 2001.

- [BdlHV08] B. Bekka, P. de la Harpe, and A. Valette. *Kazhdan's Property (T)*. New Mathematical Monographs. Cambridge University Press, 2008.
- [BF91] Marek Bożejko and Gero Fendler. Herz-schur multipliers and uniformly bounded representations of discrete groups. *Archiv der Mathematik*, 57(3):290–298, 1991.
- [BO08] Nathaniel P. Brown and Narutaka Ozawa. *C\*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.
- [CdIS15] M. Caspers and M. de la Salle. Schur and Fourier multipliers of an amenable group acting on non-commutative  $L_p$ -spaces. *Trans. Amer. Math. Soc.*, 2015.
- [CPPR15] Martijn Caspers, Javier Parcet, Mathilde Perrin, and Éric Ricard. Noncommutative de leeuw theorems. In *Forum of Mathematics, Sigma*, volume 3, page e21. Cambridge Univ Press, 2015.
- [DOS02] T.X. Duong, E.M. Ouhabaz, and A. Sikora. Plancherel-type estimates and sharp spectral multipliers. *J. Funct. Anal.*, 196(2):443–485, 2002.
- [ER00] E. G. Effros and Z.-J. Ruan. *Operator Spaces*, volume 23 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 2000.
- [Fur73] Harry Furstenberg. Boundary theory and stochastic processes on homogeneous spaces. In *Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972)*, pages 193–229. Amer. Math. Soc., Providence, R.I., 1973.
- [GPJP15] A. M. González Pérez, M. Junge, and J. Parcet. Smooth Fourier multipliers in group von Neumann algebras via Sobolev dimension. *to appear in Ann. Sci. de l'École Normale Supérieure*, 2015.
- [GPS16] A. M. González-Pérez and Mikael de la Salle. Optimal conditions for multiplier restriction results over noncommutative  $L_p$ -spaces. unpublished, 2016.
- [Haa79a] U. Haagerup. Operator-valued weights in von Neumann algebras. I. *J. Funct. Anal.*, 32(2):175–206, 1979.
- [Haa79b] U. Haagerup. Operator-valued weights in von Neumann algebras. II. *J. Funct. Anal.*, 33(3):339–361, 1979.
- [Haa85] Uffe Haagerup. Injectivity and decomposition of completely bounded maps. In *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, volume 1132 of *Lecture Notes in Math.*, pages 170–222. Springer, Berlin, 1985.
- [Har99] A. Harcharras. Fourier analysis, Schur multipliers on  $S^p$  and non-commutative  $\Lambda(p)$ -sets. *Studia Math.*, 137(3):203–260, 1999.
- [Heb92] Waldemar Hebisch. On heat kernels on Lie groups. *Math. Z.*, 210(4):593–605, 1992.
- [HR00] Nigel Higson and John Roe. Amenable group actions and the Novikov conjecture. *J. Reine Angew. Math.*, 519:143–153, 2000.
- [JMP14] M. Junge, T. Mei, and J. Parcet. Smooth Fourier multipliers on group von Neumann algebras. *Geom. Funct. Anal.*, 24(6):1913–1980, 2014.
- [JMP15] M. Junge, T. Mei, and J. Parcet. Noncommutative Riesz transforms—Dimension free bounds and Fourier multipliers. ArXiv:1407.2475, 2015.
- [Jun02] M. Junge. Doob's inequality for non-commutative martingales. *J. Reine Angew. Math.*, 549:149–190, 2002.
- [JX07] M. Junge and Q. Xu. Noncommutative maximal ergodic theorems. *J. Amer. Math. Soc.*, 20(2):385–439, 2007.
- [KW99] Eberhard Kirchberg and Simon Wassermann. Exact groups and continuous bundles of  $C^*$ -algebras. *Math. Ann.*, 315(2):169–203, 1999.
- [LdlS11] V. Lafforgue and M. de la Salle. Noncommutative  $L^p$ -spaces without the completely bounded approximation property. *Duke Math. J.*, 160(1):71–116, 10 2011.
- [NR11] S. Neuwirth and E. Ricard. Transfer of Fourier multipliers into Schur multipliers and sumsets in discrete group. *Can. J. Math.*, 63(5):1161–1187, 2011.
- [Oza00] N. Ozawa. Amenable actions and exactness for discrete groups. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(8):691–695, 2000.
- [Oza06] Narutaka Ozawa. Amenable actions and applications. In *International Congress of Mathematicians. Vol. II*, pages 1563–1580. Eur. Math. Soc., Zürich, 2006.
- [Pat88] Alan L. T. Paterson. *Amenability*, volume 29 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.



- [Ped79] G. K. Pedersen. *C\*-algebras and their automorphism groups*, volume 14 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1979.
- [Pis95a] Gilles Pisier. Multipliers and lacunary sets in non-amenable groups. *Amer. J. Math.*, 117(2):337–376, 1995.
- [Pis95b] Gilles Pisier. Regular operators between non-commutative  $L_p$ -spaces. *Bull. Sci. Math.*, 119(2):95–118, 1995.
- [Pis98] G. Pisier. Non-commutative vector valued  $L_p$ -spaces and completely  $p$ -summing maps. *Astérisque*, (247), 1998.
- [Pis03] G. Pisier. *Introduction to Operator Space Theory*, volume 294 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.
- [PX03] G. Pisier and Q. Xu. Non-commutative  $L^p$ -spaces. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1459–1517. North-Holland, Amsterdam, 2003.
- [SC02] Laurent Saloff-Coste. *Aspects of Sobolev-type inequalities*, volume 289 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002.
- [Ste70] E.M. Stein. *Topics in harmonic analysis related to the Littlewood-Paley theory*. Annals of Mathematics Studies, No. 63. Princeton University Press, Princeton, N.J., 1970.
- [Var85] V. S. Varadarajan. *Geometry of quantum theory*. Springer-Verlag, New York, second edition, 1985.
- [VSCC92] N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon. *Analysis and geometry on groups*, volume 100 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1992.
- [Zim77] Robert J. Zimmer. Hyperfinite factors and amenable ergodic actions. *Invent. Math.*, 41(1):23–31, 1977.
- [Zim78a] Robert J. Zimmer. Amenable ergodic group actions and an application to Poisson boundaries of random walks. *J. Functional Analysis*, 27(3):350–372, 1978.
- [Zim78b] Robert J. Zimmer. Amenable pairs of groups and ergodic actions and the associated von Neumann algebras. *Trans. Amer. Math. Soc.*, 243:271–286, 1978.
- [Zim78c] Robert J. Zimmer. Induced and amenable ergodic actions of Lie groups. *Ann. Sci. École Norm. Sup. (4)*, 11(3):407–428, 1978.
- [Zim84] R. J. Zimmer. *Ergodic theory and semisimple groups*, volume 81 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

KU LEUVEN CELESTIJNENLAAN 200B, 3001 LEUVEN

E-mail address: `adrian.gonzalezperez@kuleuven.be`